NCTM Regional Conference, Phoenix 2016

Group Theory Puzzles, Number Properties, and Symmetry

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Group Theory Puzzles

Copy this string of 15 letters (blocks).

RRRBRRBBRRBBBRR

BASIC RULES

- The order of the blocks matters.
- There are three “empty” strings that you may insert or delete at will:

RRR, **BBB**, **RBRB**

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RRRBRRBBRRBBBRR

RRRBRRBBRRBBBRR

BRRBBRRRR

BRRBBR

BRR RBRB BBR

BRRRBRBBBR

BBRR

BBRR RBRB

BBRRRBRB

BBBRB

RB

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RRBRBBRRRBRBBRBRRBBBBRRBRR

What do you notice? What do you wonder?

1. Can you have a string with no blocks?
2. Will you always get the same answer no matter how you simplify a string? How do you know?
3. What is the longest string in simplest form (or is there one)?
4. How many non-equivalent simplest-form strings are there?
5. Can you apply what you learn from one simplification process to help you simplify other strings more easily?
6. Do these strings actually mean anything, or is this just a puzzle?

Some strings to simplify

- BRBR

    e (the empty string)

- RBR

    BB

- BRB

    RR

- RBBR

    already simplified (but equivalent to BRRB)

How can these relationships help?

BRRBBR

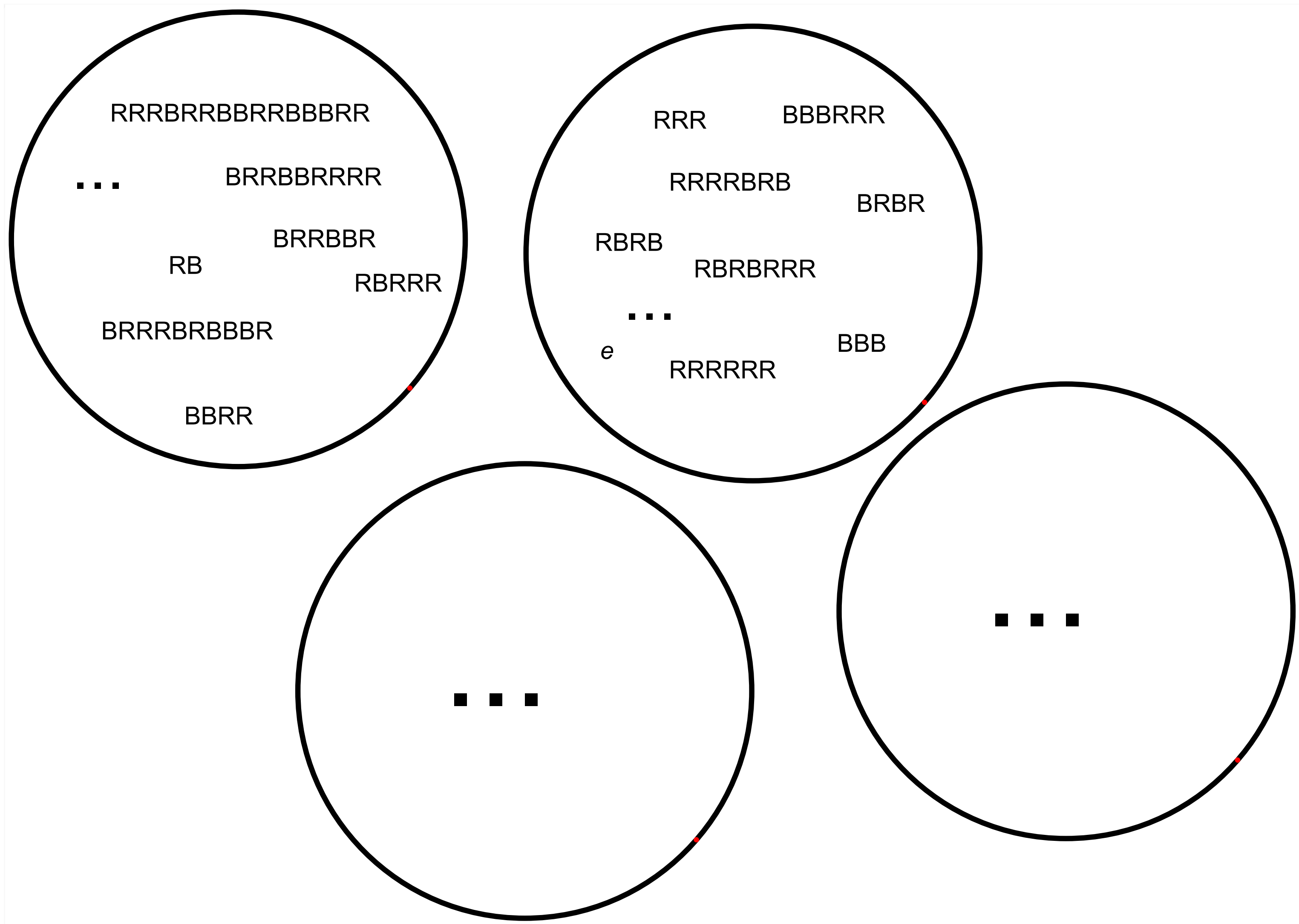
B BRB RBR R (replace RR by BRB and BB by RBR)

BBRR (remove RBRB)

BB BRBR RR (insert BRBR)

RB (remove BBB and RRR)

Families of Equivalent Strings



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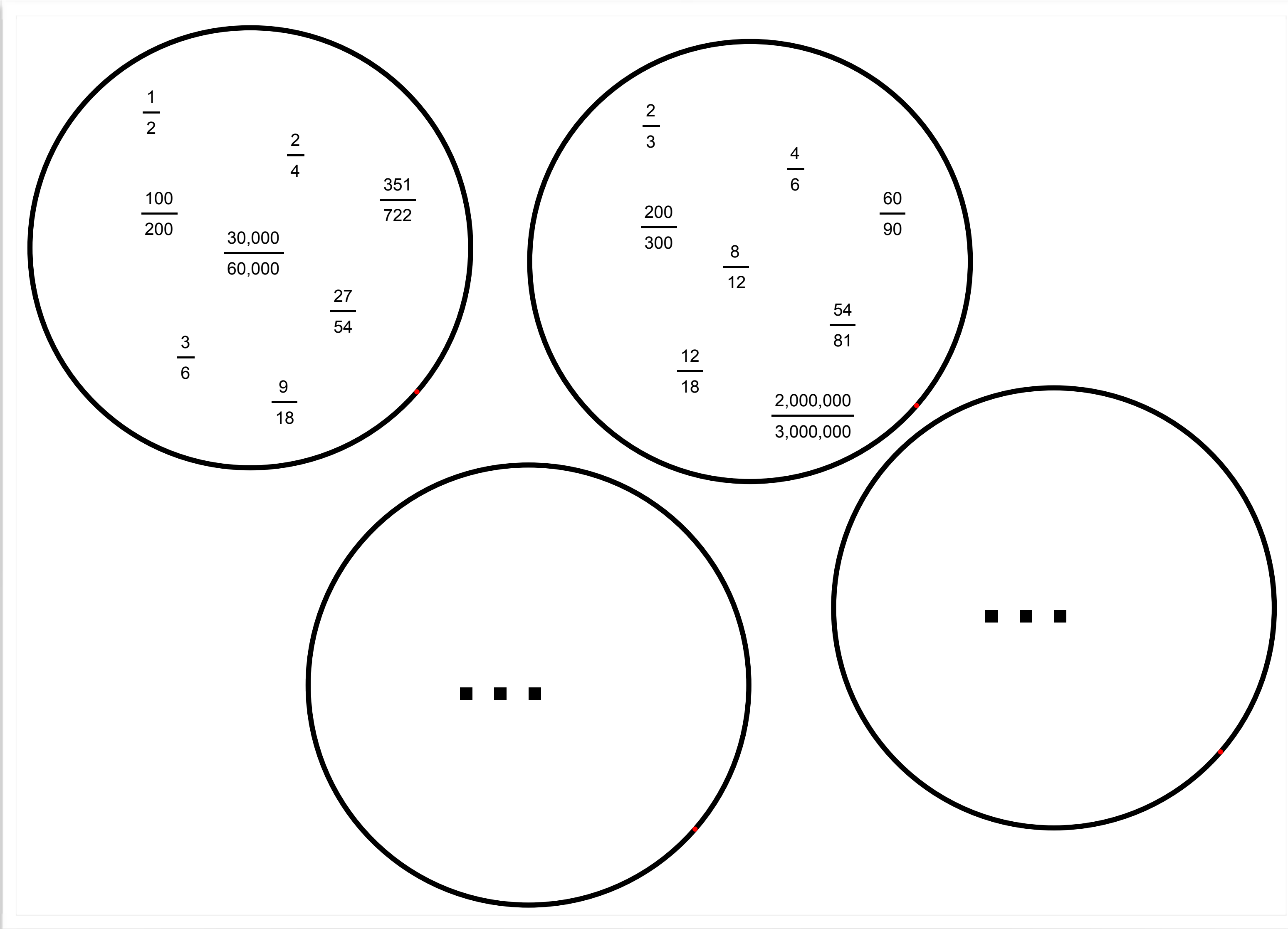
Families of Equivalent Fractions



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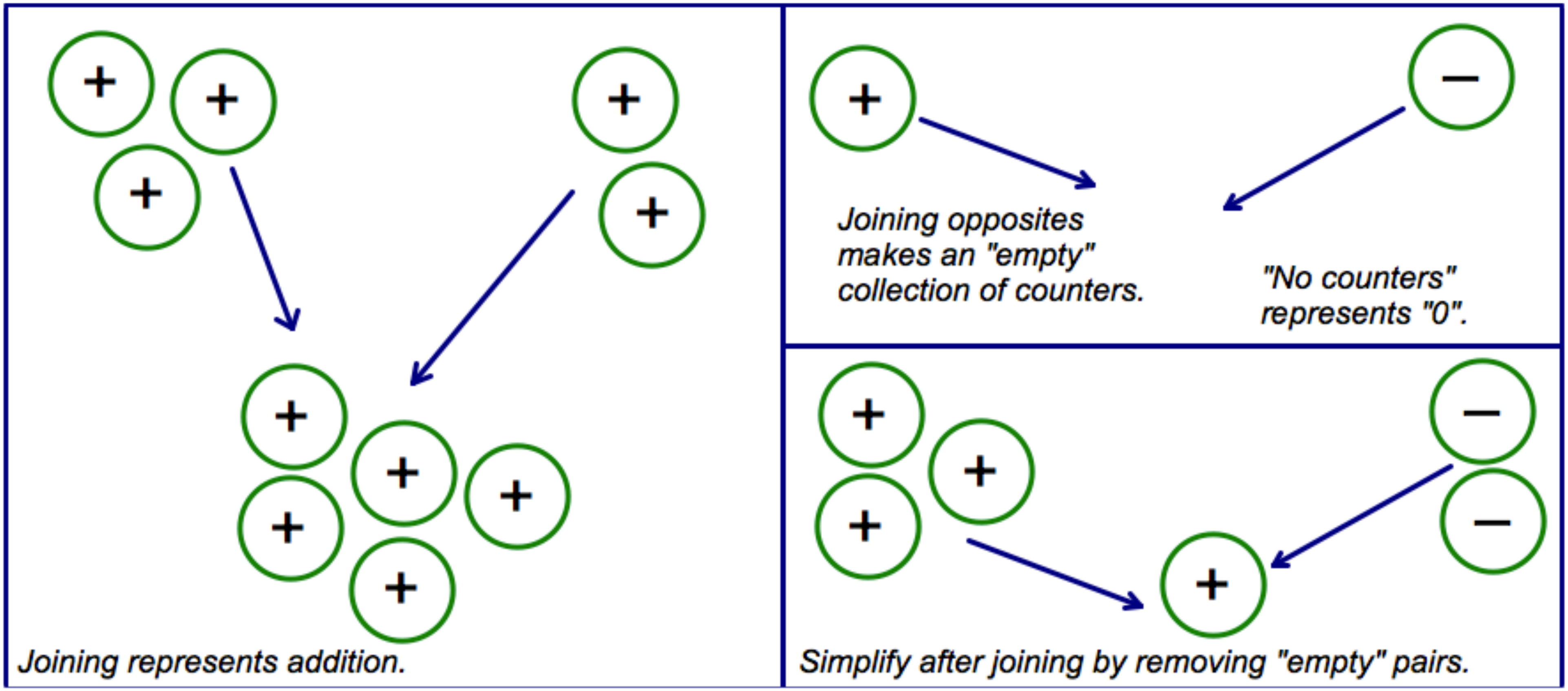
The Operation of "Joining"

- Use the symbol $\circ$.
- Example: $\mathrm{R} \circ \mathrm{RRB} = \mathrm{B}$

Connections To Middle School Math

- What middle school concepts are connected to our discussion?
  - Equivalence
  - Commutative Property
  - Associative Property
  - Identities
  - Inverses
  - Other?
- How would you do this with kids?



A Brief Detour (1)

Whole Number or Integer Counters

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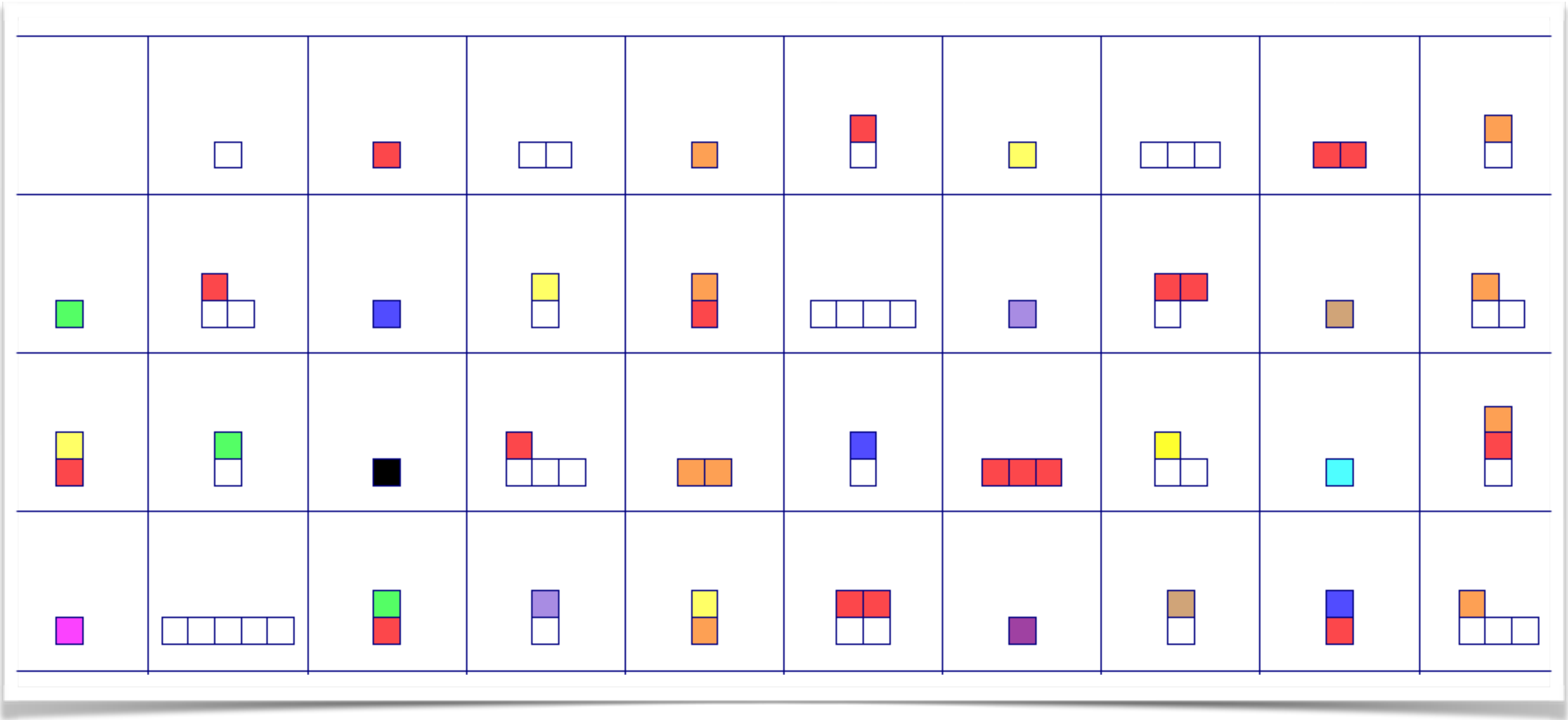

Joining represents multiplication.

A Brief Detour (2)

Prime Building Blocks

Jerry Burkhart, from, *Advance Common Core Math Explorations: Factors and Multiples*
Prufrock Press, 2014.

Is joining associative for the RB system?

YES!

- Strings that look the same are equivalent regardless of the process by which they are built (a pair at a time).
- Example: Building RRB

R	R	B	R	R	B
RR	B		R	RB	
RRB			RRB		

Is joining commutative for the RB system?

NO!

Example:

$$R \circ B \neq B \circ R$$

$$R \circ B = RB \quad \text{and} \quad B \circ R = BR$$

Does joining have an identity in the RB system?

YES!

The identity is e.

Inverses in the RB system

- The inverse of R is RR. (When you join them, you get e.)
- The inverse of B is ____ ?

BB

- The inverse of RB is ____ ?

RB

- The inverse of BR is ____?

BR

More inverses

- The inverse of RRB is ____?

BBR

- The inverse of RBB is ____?

BRR

- The inverse of RBBR is ____ ?

RBBR

- The inverse of *e* is ____ ?

e

Groups

A **group** consists of a set of objects (members) with an operation between pairs of the objects satisfying:

- The operation is ***associative***.
- There is a (single) ***identity***.
- Every member has an ***inverse***.

Note: There is actually one more condition that we will take for granted for the moment. We will return to it soon.

Back to our Questions

1. Can you have a string with no blocks?
2. Will you always get the same answer no matter how you simplify a string? How do you know?
3. What is the longest string in simplest form (or is there one)?
4. How many non-equivalent simplest-form strings are there?
5. Can you apply what you learn from one simplification process to help you simplify other strings more easily?
6. Do these strings actually mean anything, or is this just a puzzle?

How many string families are there?

- Hint: The longest simplified string is 4 letters long.

Strings to be tested

- *e*
- R B
- RR RB BB BR
- RRR RRB RBR RBB BBB BBR BRB BRR
- RRRR RRRB RRBR RRBB RBRR RBRB RBBR RBBB

  BBBB BBBR BBRB BBRR BRBB BRBR BRRB BRRR

The 12 Families of Strings

e   R B   RR RB BB BR   RRB RBB BBR BRR   RBBR

How do we know that every longer string can be simplified to one of these?

Every string of four except RBBR and BRRB can be simplified.

R**RBBR**   B**RBBR**   **RBBR**R   **RBBR**B

R**RBBR**   B**RBBR**   **RBBR**R   **RBBR**B

BRRB works similarly.

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Renaming the Group Members

e =	V1 = RB	V2 = BR	V3 = RBBR
A1 = R	B1 = BRR	C1 = RRB	D1 = BB
A2 = RR	B2 = RBB	C2 = BRR	D2 = B

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Our Group Table

$e =$ $V1 =$ RB

$V2 =$ BR $V3 =$ RBBR

$A1 =$ R $A2 =$ RR

$B1 =$ BRR $B2 =$ RBB

$C1 =$ RRB $C2 =$ BBR

$D1 =$ BB $D2 =$ B

	e	V_1	V_2	V_3	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2
e												
V_1												
V_2												
V_3												
A_1												
B_1												
C_1												
D_1												
A_2												
B_2												
C_2												
D_2												

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A4 Group Completed Table

	e	V_1	V_2	V_3	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2
e	e	V_1	V_2	V_3	A_1	B_1	C_1	D_1	A_2	B_2	C_2	D_2
V_1	V_1	e	V_3	V_2	D_1	C_1	B_1	A_1	C_2	D_2	A_2	B_2
V_2	V_2	V_3	e	V_1	B_1	A_1	D_1	C_1	D_2	C_2	B_2	A_2
V_3	V_3	V_2	V_1	e	C_1	D_1	A_1	B_1	B_2	A_2	D_2	C_2
A_1	A_1	C_1	D_1	B_1	A_2	C_2	D_2	B_2	e	V_2	V_3	V_1
B_1	B_1	D_1	C_1	A_1	D_2	B_2	A_2	C_2	V_2	e	V_1	V_3
C_1	C_1	A_1	B_1	D_1	B_2	D_2	C_2	A_2	V_3	V_1	e	V_2
D_1	D_1	B_1	A_1	C_1	C_2	A_2	B_2	D_2	V_1	V_3	V_2	e
A_2	A_2	D_2	B_2	C_2	e	V_3	V_1	V_2	A_1	D_1	B_1	C_1
B_2	B_2	C_2	A_2	D_2	V_3	e	V_2	V_1	C_1	B_1	D_1	A_1
C_2	C_2	B_2	D_2	A_2	V_1	V_2	e	V_3	D_1	A_1	C_1	B_1
D_2	D_2	A_2	C_2	B_2	V_2	V_1	V_3	e	B_1	C_1	A_1	D_1

$e =$ V1 = RB
V2 = BR V3 = RBBR
A1 = R A2 = RR
B1 = BRR B2 = RBB
C1 = RRB C2 = BBR
D1 = BB D2 = B

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Explore!

Test these on your tetrahedron:

- RBRB = BRBR = e
- RBR = BB
- BRB = RR
- RBBR = BRRB
- Calculations of your choice from the table

RRRBRRBBRRBBBRR

RRRBRRBBRRBBBRR

BRRBBRRRR

BRRBBR

BRR RBRB BBR

BRRRBRBBBR

BBRR

BBRR RBRB

BBRRRBRB

BBBRB

RB

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Did you simplify this?

RRBRBBRRRBRBBRBRRBBBBRRBRR

B

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Permutations of rgby

rgby	rgyb	rbgy	rbyg	rygb	rybg
grby	gryb	gbry	gbyr	gyrb	gybr
brgy	bryg	bgry	bgyr	byrg	bygr
yrgb	yrbg	ygrb	ygbr	ybrg	ybgr

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Even permutations of rgby

rgby	rgyb	rbgy	**rbyg**	**rygb**	rybg
grby	**gryb**	**gbry**	gbyr	gyrb	**gybr**
brgy	bryg	bgry	**bgyr**	**byrg**	bygr
yrgb	**yrbg**	**ygrb**	ygbr	ybrg	**ybgr**

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Correlation with R and B

e	rgyb	rbgy	**RR**	**R**	rybg
grby	**RBBR**	**RRB**	gbyr	gyrb	**RBB**
BBR	bryg	bgry	**BB**	**RB**	bygr
yrgb	**BRR**	**B**	ygbr	ybrg	**BR**

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Solving Equations

$$x + 7 = 10$$
$$(x + 7) + -7 = 10 + -7$$
$$x + (7 + -7) = 3$$
$$x + 0 = 3$$
$$x = 3$$

$$x \cdot 5 = 30$$
$$(x \cdot 5) \cdot \tfrac{1}{5} = 30 \cdot \tfrac{1}{5}$$
$$x \cdot (5 \cdot \tfrac{1}{5}) = 6$$
$$x \cdot 1 = 6$$
$$x = 6$$

$$x \circ B_2 = V_3$$
$$(x \circ B_2) \circ B_1 = V_3 \circ B_1$$
$$x \circ (B_2 \circ B_1) = D_1$$
$$x \circ e = D_1$$
$$x = D_1$$

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Solving Equations (2)

$$x \circ B_2 = V_3$$

$$(x \circ B_2) \circ B_1 = V_3 \circ B_1$$

$$x \circ (B_2 \circ B_1) = D_1$$

$$x \circ e = D_1$$

$$x = D_1$$

$$B_2 \circ x = V_3$$

$$B_1 \circ (B_2 \circ x) = B_1 \circ V_3$$

$$(B_1 \circ B_2) \circ x = A_1$$

$$e \circ x = A_1$$

$$x = A_1$$

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What next?

- Create new groups from other sets of empty strings.
    - RRR
    - RRR, BB, RBRB
    - RB, BR

- Makes tables for your groups.

- Look for patterns in the groups and their tables.

- Draw examples of geometric figures that have these groups.

- Look for subgroups or factor groups within your groups.

What next?

- How can you view each string as representing a *single* rotation? What is the axis of rotation? How many degrees does it turn through?

- Start with geometric shapes and find the groups that correspond to them.
  Example: Find the group for each letter of the alphabet. (Upper case works best.)

What next?

- Research connections between groups and Rubik's cube.

- Watch some related videos.
    - A TED Talk by Marcus du Sautoy about groups and symmetry:
      https://www.ted.com/talks/
      marcus_du_sautoy_symmetry_reality_s_riddle
    - Playing a Rubik’s Cube (a TedEd Talk)
      https://www.youtube.com/watch?v=FW2Hvs5WaRY

See 5280math.com for the links and for more websites.

Thank you!

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