Group Theory Puzzles
Notes for Teachers

Concepts

- Properties of numbers and operations
  - associative, commutative, identity, inverse
- Equivalence
- Patterns and structure
- Symmetry
- Permutations
- Group theory
- Solving equations

The system

The objects in the system are strings of red (R) and blue (B) blocks or letters.

There are two basic rules:

1. The order of the letters (blocks) in a string matters.
2. You may insert or delete three strings at will: RRR, BBB, RBRB

A question

What might you do with this string?

RRRBRRBBRRBBBRR

Note: Students may write the strings on paper, or you may give them physical manipulatives. I have used colored squares of tagboard or colored centimeter cubes in the past. Guide students who use manipulatives to recognize the importance of recording their simplification process!

Possible answers

You might try to simplify it (make it as short as possible by following the rules).

RRRBRRBBRRBBBRR
RRRBRRBBRRBBBRR
BRRBBRRRR
BRRBBR
BRR **RBRB** BBR
BRRRBRBBBR
BBRR
RBRB BBRR
RBRBBBRR
RBRRR
RB (simplest form)

Comments

- Every string in this list is *equivalent* to every other.
- Notice the importance of recording your work!

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Another simplification to try

RRBRBBRRRBRBBRBRRBBBBRRBRR

Possible Solution

RRBRBBRRRBRBBRBRRBBBBRRBRR
RBBRBBRBRRBRRBRR
RBBRBBRBR **BBB** RBRRBRR
RBBRBBRBRBBBRBRRBRR
RBBRBBBBRBRRBRR
RBBRBRBRRBRR
RBBRRBRR
RBBRRBR **BBB** R
RBBRRBRBBBR
RBBRBBR
RB **RRR** BRBBR
RBRRRBRBBR
RBRRBR
RBRRBR **BBB**
RBRRBRBBB
RBRBB
B (simplest form)

This is not the fastest way. Can you find a faster one?

Allow students to practice creating and simplifying their own strings. Stress the importance of recording their work.

What further questions might you ask?

Gather students' ideas for questions. If it is practical, try to follow up on these! Sample questions are listed here. We will address most of these in the course of the activity.

1. Can you have a string with no blocks?
2. Will you always get the same answer no matter how you simplify? How do you know?
3. What is the longest completely simplified string (or is there one)?
4. How many different completely simplified strings are there?
5. Can you apply what you learn from one process to simplify other strings?
6. Do these strings actually mean anything, or is it just a puzzle?

Some useful relationships

Guide students to discover the following important relationships. Later, they may use them to make the simplification process easier for other strings!

BRBR = *e*
RBR = BB
BRB = RR
RBBR is already simplified. It is equivalent to BRRB.

The joining operation

Joining ($\circ$) means to connect the two strings in the order shown in the expression. For example:

$$\text{RR} \circ \text{B} = \text{RRB}$$

An important property of joining

- *Associative* (we may assemble strings pair-wise in any way we like). Example:

$$(\text{RB} \circ \text{BRB}) \circ \text{B} = \text{RBBRB} \circ \text{B} = \text{RBBRBB}$$
$$\text{RB} \circ (\text{BRB} \circ \text{B}) = \text{RB} \circ \text{BRBB} = \text{RBBRBB}$$

Joining is *not* commutative, because the order of letters matters. For example, RB ≠ BR. This may be the first operation students have seen that is associative but not commutative! (Notice the strings have not all been simplified. Try it if you like!)

We will see more properties later.

Notes

You can easily differentiate this activity for varying needs and control the amount of time you spend on it by varying (1) the length and complexity of the strings and (2) the level and type of support that you give to students.

Before continuing, you may want to expand on the idea of equivalence. Connect it to things that students know:

- What makes two fractions equivalent?
- What makes two equations equivalent?
- Why might make two shapes equivalent? (e.g., congruence or similarity)
- What makes two strings equivalent?

Finding all families

We will use the word "family" for a collection of equivalent strings. Our task is to find all possible families in our system of strings. We will name each family by the simplest string within it. (The formal name for "family" is *equivalence class*).

Hint for students: The longest simplified string has four blocks.

A list of all strings less than or equal to four letters long

0 letters:

e

1 letter:

R	**B**

2 letters:

RR	**RB**	**BR**	**BB**

3 letters:

RRR (*e*)	BBB (*e*)
RRB	**BBR**
RBR (BB)	BRB (RR)
RBB	**BRR**

4 letters:

RRRR (R)	RBRR (BBR)	BBBB (B)	BRBB (RRB)
RRRB (B)	RBRB (*e*)	BBBR (R)	BRBR (*e*)
RRBR (RBB)	**RBBR**	BBRB (BRR)	BRRB
RRBB (BR)	RBBB (R)	BBRR (RB)	BRRR (B)

Have students test each string.

The strings followed by parentheses can be simplified as shown. The remaining strings (in **BOLD**), with the exception of BRRB, are each the name of a new family. BRRB cannot be simplified, but it belongs to the same family as RBBR.

The 12 families of strings in this system

e R B RR RB BR BB RBB BRR RRB BBR RBBR

Every string can be simplified to one—and only one—of these 12 strings.

Q: Having done all of this, how can you be sure that every string longer than 4 letters can be simplified? (If you want to explore this, begin by thinking about why every five-letter string can be simplified. See the presentation slides for more information.)

Revisiting the joining operation

- Joining is associative, but not commutative.
- The joining operation has an identity (the empty string e).
- Every family has an inverse. (For example, the inverse of R is RR.)

Rather than simply giving this information to students, ask them what they think the identity is and why. Then give them some families, and ask them what they think the inverse of each is. In other words, what string would they join to a given string to make it "disappear"? The pairs of inverses are:

e, e	RB, RB	BR, BR	RBBR, RBBR
R, RR	B, BB	RRB, BBR	RBB, BRR

When you join two inverses (in either order!), the blocks disappear, and you are left with e. Notice that each family in the top row is its own inverse!

Note: Before, continuing, you might like to emphasize what it means to have a *well-defined* operation when you have families. For example, the equation

$$\mathrm{R} \circ \mathrm{BR} = \mathrm{BB}$$

means that if you choose *any* string from the R family and join it on the left to *any* string from the BR family, you will *always* get a string from the BB family!

Definition of a *group*

A *group* consists of a set of objects and an operation on pairs of objects satisfying

1. The operation is associative.
2. The operation has an identity.
3. Every object has an inverse.

In our case, the objects are *string families*, and the operation is *joining*.

Note: Technically, there is another condition known as *closure*. This idea will make more sense after students have created a group table.

Making the group table

Give each student a copy of the handout Blank A4 Group Table (separate file).

Introducing and facilitating the task:

- Point out the bottom of the page where the families have been renamed.
- Remember that joining is not commutative. Emphasize the importance of joining in the correct order. The solutions to the table are constructed using the order left $\circ$ top. For example, to find the square in the table for $A_1 \circ V_3$, locate A_1 in the left column and V_3 on the top row.
- Try an example together with your students to get them started. (Let them do as much of the thinking as possible.) For instance:

  $A_1 \circ V_3 =$

  $R \circ RBBR =$

  $RRBBR =$

  $BRBBBR =$ (because $RR = BRB$)

  $BRR =$ (because $BBB = e$)

  B_1

  Therefore, students will enter B_1 in the square where the A_1 row and the V_3 column meet.
- Consider allowing students to collaborate. Ask each group to be responsible for one of the four-by-four squares. (I often allow students to take the table home overnight. There are usually a few students who opt to complete the entire table!)
- After students have been working for a while, ask if anyone is noticing any patterns that help them complete the table more quickly.
- Come together to share and compile everyone's results. Consider sharing the solution page with students. This will save some time, and it is beneficial for them to see the colored version anyway in order to spot patterns.

Looking for patterns!

Here are some things that students might notice:

- There are sudoku-like patterns in the table. The most important one is that every row and every column contains every member of the group exactly one time! This is a key feature of *all* groups.
- Each of the bold-outlined 4-by-4 squares contains a certain set of group members. Three of them contain only e, V_1, V_2, V_3. Three contain only A, B, C, and D with subscripts of 1. And three contain only A, B, C, and D with subscripts of 2. Furthermore, these collections also show some sudoku-like patterns!
- There are countless fascinating patterns within and between each of the 4-by-4 squares including diagonal patterns, rotations, reflections, etc.
- The identity, e, forms diagonal patterns. V_1, V_2, V_3 sometimes do the same.
- You can use the table to find inverses by looking at the rows and columns containing e.
- The set of 4-by-4 squares forms its own group if you think of each one as a single object! (A group formed in this way, by "clumping" members of a group to form a new group, is called a *factor group*.)
- There are groups inside of this group (*subgroups*). Try making tables containing only

  e, V_1 $\quad$ e, V_2 $\quad$ e, V_3
  e, A_1, A_2 $\quad$ e, B_1, B_2 $\quad$ e, C_1, C_2 $\quad$ e, D_1, D_2
  e, V_1, V_2, V_3

  Each of these eight combinations is completely self-contained. That is, if you restrict the table to only the families in the combination (using the same answers to each calculation as before), the answers will always stay within that set. For example, if you join any of e, V_1, V_2, or V_3, the answer will always be one of e, V_1, V_2, or V_3.

  This is the previously mentioned fourth property that a group must have. It is called *closure*, and it means that the answer to any calculation in the group stays within the group.

The name of the group

There are many (actually infinitely many) groups, and it can be tricky to name them and to keep track of the names. The name A_4 stands for the "*alternating* group on four elements." It is a subgroup of a larger group named S_4, which means the "*symmetric* group on four elements." In order to understand what these names mean, you need to dig quite a bit more deeply into what the groups are really all about. This is what comes next!

The tetrahedron connection

The Rs and Bs really do mean something! Perhaps the most important function of a group is to describe symmetry. We will focus on a *tetrahedron*.

Setting it up:
Show students a tetrahedron with each of the four vertices colored as follows. Place the tetrahedron on a table and color the top vertex red. Rotate it so that one vertex is facing directly away from you. Color this vertex green. Keeping the tetrahedron in the same position, color the rightmost vertex blue and the leftmost vertex yellow.

We will consider this to be the official "starting position" of the tetrahedron. Whenever the "top" vertex is red and the "away" vertex is green, it will automatically be in the starting position (if you color the vertices as described above).

Of course, you may actually choose any coloring scheme you like, but you must be consistent. It will be easiest for you to use the same scheme as I do so that the upcoming explanations are easier to read.

Suppose that *R* stands for rotating the tetrahedron 120° (one-third of a full turn) clockwise from the top beginning in the starting position. Notice that the only way you can tell that the tetrahedron has moved is by looking at the colors of the vertices. This is the reason for coloring them!

Q: What happens if you do R three times? Why is RRR equal to *e*?

B represents 120° a rotation around a different axis. We will assume that it is clockwise from the front.

Q: What happens if you do *B* three times? Why is BBB equal to *e*?

Q: What happens if you do RBRB? (The tetrahedron will return to the starting position!)

Test some other relationships:

BRBR = *e*   RBR = BB   BRB = RR   RBBR = BRRB, etc.

Use the tetrahedron to test the answers in the group table.

Look back at the solution to the very first string that you simplified.

RRRBRRBBRRBBBRR

Every string in the simplification process is equivalent. Does each string move the tetrahedron to the same position (from the starting position)?

Summarizing the experiment with the tetrahedron

The A_4 group keeps track of relationships between the symmetries of a tetrahedron. Starting from some chosen position, any string will always take you to the same position as an equivalent string.

Symmetry and permutations

Suppose that you name the position of the tetrahedron by the colors of its vertices in the order top, away, right, left. Then the initial position is represented by rgby (red, green, blue, yellow). Since *e* leaves the tetrahedron unchanged, rgby may be taken to stand for the family *e.* Each family now corresponds to a different position of the tetrahedron, which may be named using the four colors. For example, if you do the rotation R, the vertices will have the colors rygb. Thus, rygb represents the rotation R.

Q: How many ways are there to rearrange the letters r, g, y, and b? (24)

rgby	rgyb	rbgy	rbyg	rygb	rybg
grby	gryb	gbry	gbyr	gyrb	gybr
brgy	bryg	bgry	bgyr	byrg	bygr
yrgb	yrbg	ygrb	ygbr	ybrg	ybgr

Note: Each arrangement is called a *permutation* of rgyb.

Q: Why are there 24 permutations of rgby, but only 12 families of rotations?
(Because only half of the permutations relate to possible positions of the four vertices. For example, rgyb is impossible. Can you see why?)

Q: Which twelve of the permutations represent possible rotations? Which rotation does each represent?

This table summarizes the answers. Compare it to the table above. Each permutation that represents a rotation of the tetrahedron has been replaced by its matching string.

e	rgyb	rbgy	RR	R	rybg
grby	RBBR	RRB	gbyr	gyrb	RBB
BBR	bryg	bgry	BB	RB	bygr
yrgb	BRR	B	ygbr	ybrg	BR

Q: What do you observe about the connections between the permutations and the strings?

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Why the group is named A_4

A_4 stands for the "*alternating* group on 4 elements." It is a subgroup of S_4, the "*symmetric* group on 4 elements." S_4 contains all of the permutations of a collection of four objects. If we made a group table for it, we would need to do the calculations to fill in 24 • 24 = 576 boxes!

A_4 contains only half of the permutations in S_4, the so-called "even permutations." They are called even permutations because you create them using an even number of *transpositions*. A transposition is the interchange of two of the objects. For example, you can change rgby into grby as follows:

Starting position:	rgby
Transposition 1:	grby
Transposition 2:	gryb

If you look carefully back at the permutations on the previous page, you can check that the even transpositions are exactly the ones that correspond the possible rotations of the tetrahedron!

Solving equations

You can solve equations in groups just like you can with numbers. Look carefully at the properties involved. Notice the roles of associativity, inverses, and identities. Groups contain just the properties needed to carry out the solution process!

$$\begin{aligned} x+7&=10\\ (x+7)+-7&=10+-7\\ x+(7+-7)&=3\\ x+0&=3\\ x&=3 \end{aligned} \qquad \begin{aligned} x\cdot 5&=30\\ (x\cdot 5)\cdot\tfrac{1}{5}&=30\cdot\tfrac{1}{5}\\ x\cdot\left(5\cdot\tfrac{1}{5}\right)&=6\\ x\cdot 1&=6\\ x&=6 \end{aligned} \qquad \begin{aligned} x\circ B_2 &= V_3\\ (x\circ B_2)\circ B_1 &= V_3\circ B_1\\ x\circ(B_2\circ B_1)&= D_1\\ x\circ e &= D_1\\ x &= D_1 \end{aligned}$$

–7 is the *additive inverse* of 7. $\frac{1}{5}$ is the *multiplicative inverse* of 5. And B_1 is the *"join" inverse* of B_2!

Look below to see the challenges that result from the lack of the commutative property. You must be careful which side you "join" on!

$$\begin{aligned} x\circ B_2 &= V_3\\ (x\circ B_2)\circ B_1 &= V_3\circ B_1\\ x\circ(B_2\circ B_1)&= D_1\\ x\circ e &= D_1\\ x &= D_1 \end{aligned} \qquad \begin{aligned} B_2\circ x &= V_3\\ B_1\circ(B_2\circ x) &= B_1\circ V_3\\ (B_1\circ B_2)\circ x &= A_1\\ e\circ x &= A_1\\ x &= A_1 \end{aligned}$$

Exploring further

- How can you represent each family as a single rotation? What is the axis of rotation? How many degrees are in each turn?
- Choose other sets of "empty" strings, and use them to create new groups. For example:
    - {RRR}
      This group has three members. If you make its table, you will see that it is essentially the same as "addition mod 3," which basically amounts to addition on a clock that has just 3 numbers. This kind of group is called a *cyclic* group. You can easily create other cyclic groups by using different numbers of Rs.
    - {RRR, BB, RBRB}
      This is almost the same as the set we used for A_4. We have just removed one of the Bs from BBB! The group is called S_3, and it has six members. It is more complicated than the cyclic group above (for {RRR}), but not quite as complex as A_4. It is the symmetry group of an equilateral triangle. Try it!
    - {RB, BR}.
      The idea for this set of empty strings came from a student of mine. Notice that every simplified string will contain only one letter, because whenever a B and an R are next to each other, they disappear! This group has an infinite number of members. If you play around with it for a while, you will discover that the objects of this group are essentially the integers with the operation of addition!
    - What types of sets of empty strings work well? Which do not? What patterns do you see in the types of groups created? Draw examples of geometric figures that have these groups.
- Start with geometric shapes and find the groups that correspond to them. Find the group for the shape of each letter of the alphabet. Note: It works best to use upper case letters. What is the group of a shape that has no symmetry?
- Research the connection between groups and the Rubik's cube.
- Look for subgroups and factor groups inside your groups.
- Try solving equations in groups where you have two variables. For example:

  $$V_2 \circ x \circ C_1 \circ y \circ A_2 = V_3$$

  How many solutions are there? Is there a predictable relationship between *x* and *y* in each solution pair?
- Watch the videos suggested below, or find your own interesting videos about groups.

Videos and other resources about groups and their applications

- du Sautoy group theory TED talk

  du Sautoy introduces key ideas behind group theory using the group S_3, the symmetry group of an equilateral triangle. He also provides some interesting stories about the historical deveopment of group theory.

- Playing a Rubik's cube

  After working on this activity, you will probably not be surprised that group theory is central to the workings of a Rubik's cube. Can you imagine how many members this has group has?

- A book on group theory and the Rubik's cube

- The Monster group (including Conway)

  Mathematician John Conway describes the "Monster Group," a massive group that is dizzying in its complexity and intricacy and may have important things to tell us about the deep inner workings of the universe.

- The Periodic Table of Finite Simple Groups

  A many years of work, mathematicians managed to identify and categorize all groups of a certain type – finite and simple. (Simple groups are sort of the "prime numbers" of groups. They cannot be decomposed into smaller groups. This table shows all of them in an organized way that will remind you of the periodic table of elements.

- Groups of symmetries in molecules (Wikipedia)

  Symmetries in the shapes of molecules play a key role in the properties of everyday substances. Group theory helps us understand these symmetries and properties.