Noticing and Wondering Your Way to Mathematical Challenge

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Specialty: Practical resources, tools, and strategies for integrating depth and complexity into math programming and instruction for advanced learners.

Speaking, Professional Development, Consulting:

- Identifying, creating, and implementing deep, complex tasks that extend math standards for advanced learners (grades 1 – 8)
- Using formative and summative assessment with talented math students
- Motivating talented math students
- Meeting emotional needs of talented math students
- Creating a classroom culture that supports mathematical challenge
- Supporting communication between teachers, parents, administrators, and other stakeholders
- Implementing *Advanced Common Core Math Explorations* in your classroom or school (Grades 5 – 8)
- Defining and applying appropriate uses of math acceleration

Resources; instructional modeling, observation, and reflection; interactive workshops; presentations; individual and team consulting

Publications:

Advanced Common Core Math Explorations: Factors and Multiples (2014)
Advanced Common Core Math Explorations: Numbers and Operations (2014)
Advanced Common Core Math Explorations: Fractions (2014)
Advanced Common Core Math Explorations: Measurement and Polygons (2015)

coming in 2016:
Advanced Common Core Math Explorations: Ratios, Similarity, and Proportions
Advanced Common Core Math Explorations: Probability and Statistics

all books published by Prufrock Press
www.prufrock.com

Background:

- Experience: 20 years of teaching talented math students
- Experience: 7 years professional development/coaching
- Education: M.A., Mathematics: M.A.T., Math Education

$$0 = 4 - 4$$

$$1 = 4 - 3$$

$$2 = 4 - 2$$

$$3 = 4 - 1$$

vocabulary: *difference* = *minuend* – *subtrahend*

I wonder if it matters that the difference is on the left side of the "=" sign.

I notice that the minuend does not change.

I notice that the subtrahend decreases by 1 each time.

I notice that the difference increases by 1 each time.

I notice that when the minuend stays the same and the subtrahend decreases, the difference increases.

I wonder if this is still true no matter what the minuend is.

I wonder what happens to the difference if the minuend increases and the subtrahend stays the same.

I wonder what happens to the difference if both subtrahend and minuend increase by 1.

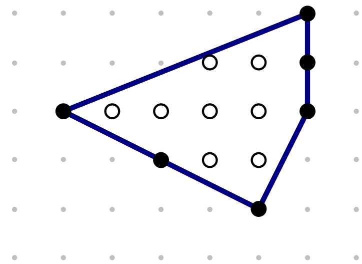
I wonder what happens to the difference when the subtrahend and minuend increase or decrease by other amounts.

I wonder what happens if the subtrahend is greater than the minuend.

I wonder what happens if the subtrahend is negative.

I wonder if I can fill equations *in between* the equations that were given.

I wonder if these equations would follow the same kinds of patterns.



I notice that the black dots are on the border (*boundary*) of the polygon.

I notice that the open dots are on the inside (in the *interior*) of the polygon.

I notice that there are 6 points on the boundary.

I notice that there are 8 points in the interior.

I wonder how I can be sure a point is on the boundary when it looks very close on paper.

I wonder if there are other polygons that have the same number of boundary and interior points as this one?

I wonder how many polygons have this number of boundary and interior points.

I wonder if there are polygons that have no interior points.

I wonder if I can make very large polygons that have no (or a small number of) interior points.

I wonder if I can make polygons that have any number of boundary and interior points.

I wonder if I can make this polygons' area greater without changing number of interior points.

I wonder what happens to the number of boundary and interior points when I drag one of the boundary points 1 unit of the right (left, up or down).

I wonder if polygons with larger areas always have more interior points.

I wonder if polygons with larger areas always have more boundary points.

I wonder if I can use the number of boundary points and interior points to predict the area?

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x 4 ÷ 8

64					2

I wonder why "x 4" and "÷ 8" are written above the circles.

I wonder if I should put numbers in the other circles.

I notice that the numbers given in the problem are all even.

I notice that the (whole) numbers I put in the circles are always even.

I wonder if I can find a way to multiply or divide with large numbers before I've been taught how to do this.

I wonder if the circles can have fractions or decimals in them.

I wonder if there is more than one solution.

I wonder if the solutions have anything in common.

I notice that in order to make 64 so much smaller, I have to divide by 8 more often than I multiply by 4.

I wonder how I can tell when I have found all of the solutions.

I notice that when I put a fraction in a circle, the denominator is always even.

I wonder if there is a way to do the problem backward (turning 2 into 64).

I wonder if there is connection between the solutions to the backward problem and the original problem.

I wonder if the problem can be solved if I reverse the operations (to ÷ 4 and x 8).

I wonder if I can create problems like this that have no solution.

I wonder if I can create other problems like this that have as many solutions as this one does.

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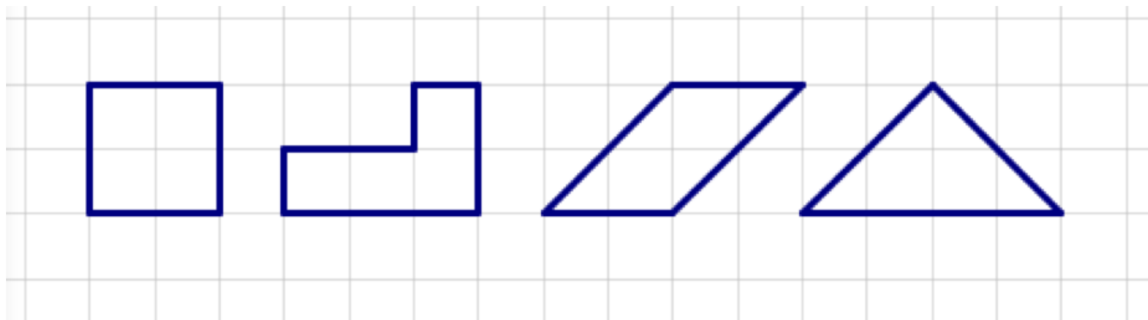

I wonder if these polygons all have something in common.

I wonder if the polygons are in some special order from left to right.

I notice that the first two polygons have only vertical and horizontal sides.

I notice that the first two polygons are built from four squares.

I *wonder* how many different polygons I can make by joining four squares.

I wonder what would happen if I joined the squares so that the edges did not match.

I notice that all of the polygons have an area of 4 square units.

I wonder if a shape is a polygon if the squares that make it touch only at the corners (*vertices*).

I wonder how many polygons (or triangles, quadrilaterals, pentagons, etc.) I can make that have an area of 4 square units.

I notice that the polygons do not all have the same perimeter.

I wonder how great I can make the perimeter of a polygon whose area is 4 square units.

I notice that the "diagonal" of a square is longer than its side.

I notice that the lengths of the "diagonal" sides are not whole numbers of units.

I notice that all of the polygons are built from squares and half-squares.

I wonder if I can make polygons with an area of 4 square units that are not made of only squares are half-squares.

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·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

I notice that every grey number is a "whole number times itself."

I notice that you increase grey numbers by successive odd numbers to get the next grey number. 1 + 3 = 4   4 + 5 = 9   9 + 7 = 16   16 + 9 = 25, etc.

I notice that you do the same thing with the yellow numbers, but you start by adding 5.

I wonder if these patterns continue if you extend the multiplication table.

I notice every yellow number is 1 less than the grey number to its lower-left.

I wonder why this happens. (Try making arrays and moving dots to turn one multiplication expression into another; for example to turn 5 x 5 into 4 x 6).

I notice that I can use this pattern to do some calculations (like 19 x 21 or 21 x 21) in my head.

I notice that there is another diagonal identical to the yellow one to the left and below the grey numbers.

I notice that the next diagonal (2 above and to the right of the grey square) is always 4 less than the number in the grey square.

I notice that this kind of pattern continues for diagonals that are farther away. They are 1, 4, 9, 16, etc. less than the number in the grey square.

I notice that these numbers are the same as the grey numbers!

I wonder why this happens.

I wonder if there are other patterns in the multiplication table.

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8	1	6
3	5	7
4	9	2

3	3	3	3
			3
			3
			3
			3

For the original square:

I notice that all of the whole numbers from 1 through 9 are used.

I notice that the middle-sized number (which is the mean and the median in this case!) is in the middle of the square.

I notice that one diagonal contains the consecutive numbers 4, 5, and 6.

I notice that the smaller and larger numbers seem to be spread out in a balanced way around the square.

I notice that the rows, columns, and diagonals all have a sum of 15. (This is called a *magic square.*)

I wonder if this is the only way to make a magic square that has sums of 15.

I wonder if there is a pattern to how the numbers are arranged—something that could be used to create larger magic squares (with more rows and columns).

For the square with sums of 3:

I wonder if all nine numbers have to be different.

I wonder if I negative numbers are allowed.

I wonder if decimals or fractions are allowed.

I wonder if 1 half and 2 fourths are the same number.

I wonder if the original square can help me find an answer for the square with sums of 3.

I notice that I can divide all of the numbers in the original magic square by 5.

I wonder if I can make magic squares that have any sum I want.

I wonder if I can add, subtract, multiply, or divide two magic squares. I wonder if I can get new magic squares this way.

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$$\frac{3}{5} \quad \frac{11}{18} \quad \frac{8}{13} \quad \frac{13}{21} \quad \frac{5}{8}$$

I notice that I can make a lot of addition or subtraction equations with the numerators and denominators of these fractions.

I notice that the smallest numerators and denominators are on the ends of the list.

I notice that the fractions are very close in size!

I wonder if there is a pattern to how the list was created.

I wonder if I can extend the pattern to the left or the right.

I notice that the fractions increase from left to right.

I wonder why the fractions increase from left to right.

I wonder if there is a pattern to how much each fraction increases as I go from left to right.

I wonder if the fractions will keep increasing from left to right if I extend the list.

I wonder what will happen if I remove the second and fourth fractions from the list.

I wonder if I could create other lists using this same pattern beginning with different fractions on the left and right.

I wonder if the fractions in these lists will always increase from left to right.

I wonder what will happen if I begin with the greater fraction on the left.

I wonder what will happen if I begin with equivalent fractions on the left and right.

I wonder if anything interesting would happen if I used multiplication (or another operation) on the numerators and denominators to create the fractions in between.

adapted from *Advanced Common Core Math Explorations: Fractions*
by Jerry Burkhart, Prufrock Press: 2014
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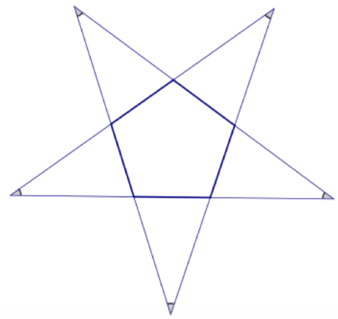

I notice a regular pentagon in the center of the star.

I notice five isosceles triangles outside the pentagon.

I wonder how many triangles are in the picture?

I notice five congruent kites.

I notice that the star has rotational symmetry.

I wonder why the star-tip angles are marked.

I wonder that the measure of each star-tip angle is.

I wonder what the ratios of some of the side lengths in the star are.

I notice that you could continue the design outward and inward to create more similar stars and similar pentagons.

I wonder what the measures of the star-tip angles would be if you made stars from regular hexagons, heptagons, octagons, etc.

I wonder if the measures of these star-tip angles (or their sums) form a pattern.

I wonder if there is a formula that will predict the measures of the star-tip angles from the number of sides of the regular polygon in the center.

I wonder what happens if I use my formula on equilateral triangles and squares.

I notice that regular polygons with 7 or more sides make multiple layers of stars.

I wonder what the measures of the star-tip angles are for the stars in different layers.

I wonder under what circumstances you can make stars from irregular polygons.

adapted from *Advanced Common Core Math Explorations: Measurement and Polygons*
by Jerry Burkhart, Prufrock Press: 2015
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Bristlecone School District data

Mountain Heights Middle School:
51,600 square feet of floor space
470 students

North Star Middle School:
118,300 square feet of floor space
725 students

I wonder why the school district collected this data.

I wonder how it would feel to be at each school.

I notice that when the population density (number of people per unit area) is greater, the school is more crowded.

I notice that when the area per person is greater, the school is less crowded.

I notice that the area per person is the reciprocal of the population density.

I wonder how to make Mountain Heights Middle School less crowded.

I wonder how the area per person compares at different schools in our district.

I wonder what the area per person is in our classroom.

I wonder how much it would cost to expand the Mountain Heights building.

I wonder what the district can do if they don't have money for building projects.

I notice you could improve the crowding by moving students.

I wonder how many students would have to move to solve the problem.

I wonder if I can find the number of students without using trial and error.

I wonder if it would help to imagine putting the two schools together.

I wonder if I could use algebra to solve the overcrowding problem.

I wonder if there are other things that might make a school building feel crowded?

adapted from *Advanced Common Core Math Explorations: Ratios, Proportions, and Similarity*
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$$2+2 \qquad 2\cdot 2$$

$$-1+\frac{1}{2} \qquad -1\cdot\frac{1}{2}$$

I notice that the sum and the product of two numbers can sometimes be equal.

I notice that the two numbers may be the same or different.

I wonder if there are other pairs of numbers whose sum and product are equal.

I wonder if every number has a partner that makes the sum and product equal (a *sum-product partner*).

I wonder if there is a fast way to find a number's sum-product partner.

I wonder if there is a fast way to predict the sum and product of a number and its partner.

I wonder if there is a formula for sum-product partners.

I wonder if my formula (or pattern) makes each number's partner's partner equal to the number. (Wow – you may have to read this one a few times! Putting it another way: Does my formula make the numbers partners of *each other*?)

I wonder if I can prove this using algebra.

I wonder if I can find sum-product partners for irrational numbers like π and $\sqrt{2}$.

I wonder if numbers can be put into pairs whose sum and product are the opposite of each other.

I wonder if numbers can be put into pairs whose sum and product differ by 1?

I wonder if there is such a thing as a difference-quotient partner.

I wonder if there are any connections between sum-product partners (pairs) and reciprocal pairs.

I wonder if sum-product pairs are inverses for some operation (in the same way that opposites are inverses for addition, and reciprocals are inverses for multiplication).

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Number	**Code**	**Number**	**Code**
0	none	8	3
1	0	9	20
2	1	10	101
3	10	11	10000
4	2	12	12
5	100	13	100000
6	11	14	1001
7	1000	15	110

I wonder if there are patterns going down the code columns.

I wonder if there are patterns going across between the numbers and the codes.

I notice that all of the codes contain only the digits 0, 1, 2, and 3.

I wonder if the codes for all of the numbers contain only the digits 0, 1, 2, and 3.

I notice that a lot of the codes look like powers of ten (1, 10, 100, 1000, etc.)

I wonder which numbers have these codes.

I notice a lot of patterns that appear to get started and then fall apart.

I notice that the numbers for the codes 0, 1, 2, and 3 are 1, 2, 4, 8.

I wonder if 16 has a code of 4.

I notice connections between codes and exponential expressions.

I wonder if there is a pattern when numbers triple?

I notice a connection between the codes for the numbers 2, 3, and 6.

I wonder if it is better to think of the codes as numbers or as strings of digits.

I wonder why 0 cannot be put into code.

I wonder if all counting numbers can be put into code.

I wonder if fractions or negative numbers can be put into code.

I notice a connection between codes and prime factorizations.

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