Deep Algebra Projects: Algebra 1 / Algebra 2
Go with the Flow

Topics

- Solving systems of linear equations (numerically and algebraically)
- Dependent and independent systems of equations; free variables
- Mathematical models

The Go with the Flow project involves analyzing traffic flow in a variety of situations that become more complex as the activity progresses. It deepens and extends students' *conceptual* understanding of systems of linear equations by focusing on systems of many equations that have multiple solutions but that can be solved (at least in part) numerically.

Students will have opportunities to practice the traditional substitution and elimination (addition) algorithms. However, rather than using them only to find numeric solutions, students will spend much of their time using these processes to discover and describe patterns, predict the number of solutions, and interpret solutions in real-world contexts.

Unlike most extension projects in this series, individual problems do not have their own Conversation Starters pages. Instead, there is a set of General Conversation Starters (see the next page) to use with all of the problems. You and your students may decide which questions best apply to a given problem.

The Solutions in this project are quite a bit longer than usual—generally three to six pages—mainly because there are so many possible questions to explore. (Even so, you and your students may think of questions to ask that are not included!) I recommend preparing for the project by reading the Solutions pages for Problem #1 thoroughly. These pages address most of the important general ideas in the project in the context of a situation that is less complex than in later problems.

General Conversation Starters

Ask and answer questions like these in order to analyze traffic flow in depth. Choose the questions that seem to apply best to the problem you are working on.

Do the diagrams appear to represent realistic situations? Why or why not?
How can you model the traffic flow using a system of equations?
Do you see any patterns in the equations? If so, how would you describe the patterns?

Is there an infinite number of solutions? If so:

- How can you describe the solutions numerically?
- How can describe the solutions algebraically?
- How could you have predicted that there would be infinitely many solutions?
- What is the significance of an infinite number of solutions for traffic flow?

Are any equations in the system automatically true when others are true? If so:

- How many? Which ones?
- How can you tell numerically?
- How can you prove this algebraically?

What is the possible range of values for each variable? How can you tell?
What is the significance of these ranges for traffic flow?

Which streets could be closed for construction with the least impact on traffic flow?
How could you manage the flow in these cases?
Are there some streets that could be closed *completely* without seriously impacting traffic flow? How can you tell?

Is it practical to imagine the detailed routes that drivers might take through an intersection or set of intersections?
Does this help you understand the traffic flow better? If not, why not? If so, how?

What is the usefulness of having knowledge of the traffic flow on different roads?
What limitations do you see in the mathematical models that you create? Can you think of ways to improve them?

Stage 1

In Problems #1 and #2, students use systems of linear equations to model traffic flow in roundabouts. In order to anticipate the types of questions and thinking involved, I recommend that you read the General Conversation Starters (previous page) and the Solutions for Problem #1 in advance.

Begin Problem #1 by showing students the picture of the roundabout without the directions. Ask them what they notice and wonder. Use the opportunity to generate questions for clarification and exploration. After the conversation, give each student a copy of the Problem Page and the General Conversation Starters page. Depending on your learning objectives for students, you may use the directions as they are, modify them, select specific questions in the Conversation Starters for students to answer, or offer students some choice in the questions that they pursue.

After students have worked for a while, you may need to guide them toward making tables and/or expressing some of their ideas and solutions algebraically. In particular, they may need to be encouraged to (1) determine minimum or maximum possible values for each variable, (2) express their solutions using variables, (3) think about what is happening when $w \neq 0$, and (4) try to understand why one of the four equations automatically appears to be satisfied once the other three are satisfied.

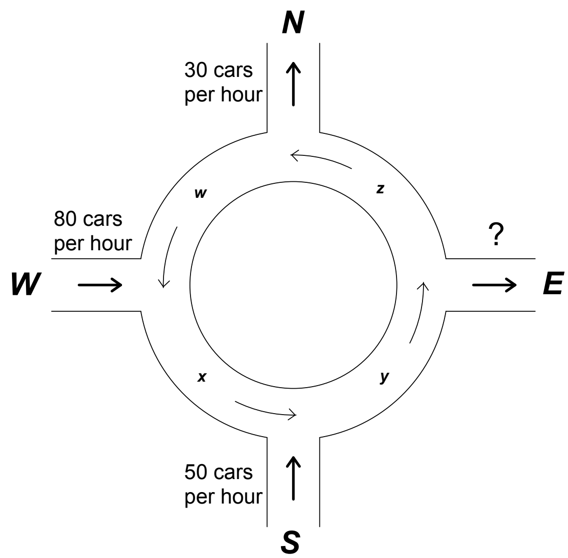
What students should know

- Use the substitution or elimination algorithms to solve systems of two linear equations in two variables. (Students may learn these skills concurrently with their work on Stage 1.)

What students will learn

- Recognize systems of linear equations that have multiple solutions.
- Make connections between numeric and algebraic methods of solving equations.
- Examine patterns in solutions to systems of linear equations.
- Find maximum and minimum values for solutions in real-world contexts.
- Describe solutions efficiently when there is a large number of them.
- Recognize dependent sets of equations (and prove their dependence).
- Create, analyze, and interpret mathematical models.

Problem #1



Directions

- Explain what must happen in the roundabout overall and at each of the four intersections over the course of an hour so that traffic doesn't pile up.
- Find the value of the question mark. Explain your thinking.
- Guess what w, x, y and z represent, and find their values. Explain your thinking.
- Think of other questions to ask. Answer them.

Diving Deeper

- Suppose that traffic in the roundabout flowed clockwise. Predict how this would affect your answers to the questions. Use calculations to test your predictions.

Solutions for #1

Conditions for traffic not to pile up

In the whole roundabout: The number of cars entering the roundabout over the course of the hour should equal the number of cars exiting the roundabout.

At each entry/exit point of the roundabout: The number of cars entering each intersection over the course of the hour should equal the number of cars exiting that intersection.

The value of the question mark

The value of question mark is 100 cars per hour.

There are $50 + 80 = 130$ cars entering the roundabout on the west and south sides of the roundabout in one hour. There are 30 cars exiting on the north. Because the number of cars entering and exiting the roundabout over the course of an hour should be equal:

$$50 + 80 = 30 + ?$$

There are $130 - 30 = 100$ cars exiting on the east where the "?" is shown.

The meanings of w, x, y and z

$x, y, z,$ *and* w represent the number of cars per hour passing through each of the four regions sections of the roundabout.

The values of w, x, y and z (using a table)

Students might begin by guessing values for a variable and watching the effects. Eventually, they may see the benefits of organizing their results in a table.

w	*x*	*y*	*z*
0	80	130	30
10	90	140	40
20	100	150	50
30	110	160	60

As the table shows, there are many solutions. Each solution represents a situation in which the number of cars entering and exiting each intersection is equal over the course of an hour. The solutions show clear patterns. (These are discussed later in these Solutions.)

Notes: Students' tables may vary. Ideally, they will organize the numbers, but they may choose to show values in other increments such as 1 or 5. It is also likely that they will discover that some values don't work. For example, if they choose $x = 70$ or $y = 120$, they will discover that $w = -10$, which does not make sense.

The values of w, x, y, and z (using variables and equations)

Students may represent the situation using algebraic equations. For example, if the left side of each equation shows cars entering an intersection and the right side shows cars leaving the intersection:

South	$x + 50 = y$
East	$y = 100 + z$
North	$z = 30 + w$
West	$w + 80 = x$

Some students may choose to write the equations differently—for example, by placing variables on the left and numbers on the right.

South	$y - x = 50$
East	$y - z = 100$
North	$z - w = 30$
West	$x - w = 80$

The second way of writing the equations highlights certain patterns.

- Each variable appears in exactly two equations.
- Every equation involves subtraction.
- In the first terms, y appears twice, but w never shows up.
- In the second terms, w appears twice, but y never shows up.

Regardless of how you write the system, you can quickly obtain the values in the table by substituting different numbers into the equations. Furthermore, it is easy to check that the equations fit the table, because they describe the relationship between each pair of variables for every row. For example, the difference between x and y is always 50 in the table as the equation $y - x = 50$ states.

Is the situation realistic?

Students may have some doubts. For example, they may wonder if a roundabout is needed for two one-way streets. They may also try to decide if the number of cars per hour is reasonable. (It probably is.)

In order to dig deeper into this issue and others, think of your own questions to ask, or consider some of the ideas and questions in the General Conversation Starters.

Patterns and minimum values

In the table, the values of w, x, y, *and* z increase together at a constant rate. Any time one of the variables increases by some value, the others increase by the same amount. There is no mathematical limit to how large the values can become! The minimum value for each variable ($x = 80, y = 130, z = 30$) occurs when $w = 0$, because any other value leads to a negative number of cars per hour for some variable, which makes no sense.

The minimum values are not as immediately apparent from the equations as from the table. However, it is clear from the North equation that z must be greater than or equal to 30. From this (by substituting $z = 30$ into the East equation) it follows that $y \geq 130$, $x \geq 80$, and $w \geq 0$. Of course, none of the variables can be negative, but you may need to look a little more closely to see that w can *equal* 0. (Notice that w is the only variable that never appears as the first term in any equation.)

The equations form a sort of chain. Because each equation contains only two variables, once you choose one value, you may use an equation to determine the next value, another to choose the next, etc. until you have found all four values. Interestingly, since any set of three equations contains all four variables, the fourth equation is not needed. More importantly, it appears that the fourth equation is *automatically* satisfied when the other three are!

Proving that the fourth equation is automatically satisfied

Any single equation is automatically satisfied when the other three are. That is, every equation is a consequence of the other three. To prove this, you may use algebraic methods to derive any equation from the remaining ones. For example, consider the following method for deriving the West equation by substitution.

Begin with the South equation.

$$y - x = 50$$

Use the East equation to obtain an expression for y.

$$y = 100 + z$$

Substitute this expression for y into the South equation.

$$100 + z - x = 50$$

Use the North equation to obtain an expression for z.

$$z = 30 + w$$

Substitute this expression for z into the previous equation.

$$100 + 30 + w - x = 50$$

Simplify and rearrange to obtain the West equation.

$$130 + w - x = 50$$
$$80 + w - x = 0$$
$$x - w = 80$$

Students could use the elimination (addition) method as well. For example:

Add the East equation to the opposite of the South equation.

South: $-y + x = -50$
East: $y - z = 100$
Sum: $x - z = 50$

Add the North equation to the Sum. (The Sum is rewritten to align variables.)

Sum: $-z + x = 50$
North: $z - w = 30$
New Sum: $x - w = 80$

The New Sum is the West equation!

Important notes

- Encourage students to experiment! They may use substitution or elimination with *any* of the three equations to produce the fourth.
- The process above shows that substitution and elimination techniques are useful for purposes other than obtaining numeric solutions. They may also help with finding new relationships and analyzing real-world and mathematical situations.
- Because at least one equation depends on (may be derived from) the other three, the original set of four equations is called *dependent*. Discarding a dependent equation leaves you with a set of three *independent* equations and has no effect on the solutions!
- Since the number of variables (4) is greater than the number of independent equations (3), $4 - 3 = 1$ of the variables (it does not matter which) is *free* to take on a range of values, resulting in an infinite number of solutions. However, once you choose this value, the others are determined. (Mathematically, the free variable could take on *any* value, but the *realistic* set of values is restricted by the real-world condition that only whole number values make sense.)

The significance of the infinite number of solutions for traffic flow

From a practical perspective, it is a good thing that this problem has many solutions! If there were only one possible set of values for $w, x, y,$ *and* z**,** there would be only one traffic configuration that would allow for smooth traffic flow. Anything else might cause traffic to jam up!

imagining the specific routes that drivers might take

The system of equations shown in these solutions does not take account of the actual routes that drivers take through the roundabout. It can be interesting to see if paying attention to this issue gives any new insights into the real-world scenario.

Routes that drivers might take

- Enter from the south. Drive along the y region. Exit to the east.
- Enter from the south. Drive along the y and z regions. Exit to the north.
- Enter from the west. Drive along the x and y regions. Exit to the east.
- Enter from the west. Drive along the x, y, and z regions. Exit to the north.

Notice that none of these options involves driving in the w region! This seems like a good thing, because the model showed that w has the smallest values of any variable and can even be equal to 0. In fact, the only people driving in this part of the roundabout would be those who miss their turnoff and go around an extra time.

Interestingly, any time that a driver does this, x, y, and z will be one greater than they would have been had the driver turned off the first time around. This matches well with the behavior that you see in the table: whenever w increases by 1, the other three variables do as well.

Since drivers will probably not miss their turnoffs too often, the vehicle counts during any given hour are likely to appear in a row of the table that has a small value of w. The rows containing very large numbers are not likely to occur even though they are possible in the model. This is fortunate! If too many drivers went around and around, the total volume of traffic would become too large to manage.

The discussion above illustrates the point that mathematical models may not include all of the information needed for a complete understanding of a real-world situation. In our case, the system of equations by itself did not tell us everything that we might have wanted to know. Learning to create models that do manage to capture the important information without being unnecessarily complex is one of the interesting challenges in using mathematics to analyze real-world situations!

Describing the infinite number of solutions

People often use a set of algebraic expressions to describe an infinite number of solutions. For example, you may write the solution to this problem as:

$$(w, w+80, w+130, w+30);\ w \text{ is a whole number}$$

where the ordered quadruple represents (w, x, y, z).

In the example above, w is the free variable, but you may choose any of the four variables. For example, if you choose x, you will get:

$$(x-80, x\ , x+50, x-50);\ x \text{ is an integer greater than or equal to 80.}$$

If you choose different values for w and x (according to the restrictions listed after the semicolons), you will find that both representations describe exactly the same set of solutions.

Note: You may also describe the set of solutions using a table like the one on the first page of these Solutions, but this approach is generally less precise. You may need to include an explanation that describes the patterns.

Managing traffic flow during construction

During construction, it may be possible to work on one portion of the roundabout at a time. The w-section will be the easiest, because it is used only by people who miss their turnoff. You might decide just to put up a sign warning drivers that this section of the roundabout is closed.

Travelers entering from the west and exiting north need to use the z-section. You may need to provide a detour to offer them an alternate route. If the roundabout has two lanes, you might close just one of them at a time. The z-section should not be too difficult to manage because it carries much less traffic than the x- and y-sections.

Since sections x and y carry quite a lot of traffic, you may need to divert traffic using a detour and possibly even close the entire roundabout while work is done on them.

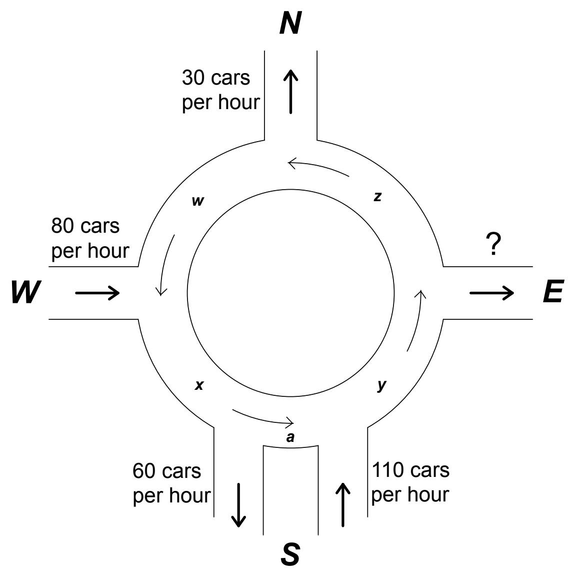
Possible limitations of the model

Although the model does a good job of looking at the big picture of traffic flow in the roundabout, it misses certain details. For example, the hourly averages in the model do not take account of the flow at specific times. It may be that there is very little traffic during part of an hour and spikes in traffic volume at other times. If these volumes are high enough, they might cause traffic to slow down and stop for brief periods. Measurements used in the model will also be imperfect. Some cars may not get counted if they go around many times, get stalled in the intersection, or happen to enter or exit just before or after a measurement is taken. However, these discrepancies are likely to be too small to affect the conclusions drawn from the model.

Students may also have noticed that the equations by themselves do not account for specific routes that drivers take through the roundabout. In Problems #1 and #2, paying attention to these routes made a difference in understanding the traffic flow. In other cases, it may make less of a difference—or it may simply be too difficult to track this information.

Finally, many factors that may be important to consider are not included in the model. Traffic engineers may need to take account of the types and sizes of vehicles, typical speeds in the roundabout, drivers' lines of sight, changes in traffic volume at different times of the day, etc.

Problem #2



Directions

- Figure out as much as you can about the traffic flow in this roundabout.
- Compare and contrast the situation and your answers with Problem #1.
- Explain what it might mean to exclude the variable a from this model.

Diving Deeper

- Suppose that all of the streets above were two-way. How many equations would there be? How many variables would each equation have? What additional information would you need in order to understand the traffic flow? How many path options would each driver have in and out of the roundabout?

Solutions for #2

The value of the question mark

The value of question mark is still 100 cars per hour, because the North and West intersections have the same traffic flow as in Problem #1, and the *net* inflow of traffic from the South intersections is still 50 cars per hour. However, the equation in this case,

$$110 + 80 = 60 + 30 + ?$$

shows a greater number of cars entering and exiting the roundabout each hour than in Problem #1.

The values of $w, x, y, z,$ and a (using a table)

By guessing values for a variable and watching the effects, students should discover a table that looks the same as the table for Problem #1 except that there is a new column for a with its own set of values.

w	*x*	*y*	*z*	*a*
0	80	130	30	20
10	90	140	40	30
20	100	150	50	40
30	110	160	60	50

Again, there is an infinite number of solutions, and they display clear patterns.

The values of $w, x, y,$ and z (using variables and equations)

The algebraic equations are similar to those in Problem #1. In fact, the equations for East, North, and West are identical. The South equation has been replaced by two separate equations.

South (entering)	$a + 110 = y$
East	$y = 100 + z$
North	$z = 30 + w$
West	$w + 80 = x$
South (exiting)	$x = 60 + a$

If you arrange the equations by placing variables on the left and numbers on the right, you obtain

South (entering)	$y - a = 110$
East	$y - z = 100$
North	$z - w = 30$
West	$x - w = 80$
South (exiting)	$x - a = 60$

As before, the second way of writing the equations highlights certain patterns.

- Each variable appears in exactly two equations.
- Every equation involves subtraction.
- In the first term, x and y appear twice, but w and a never show up.
- In the second term, a and w appear twice, but x and y never show up.
- z is the only variable that appears once each in the first and second terms.

Again, the equations accurately describe the relationship between each pair of variables for every row. For example, the difference between x and a is always 60 in the table, which agrees with the equation $x - a = 60$.

Patterns and minimum values

In the table, the values of $w, x, y, z,$ and a increase together at a constant rate. Specifically, any time that one of the variables increases by some value, the others increase by the same amount. There is no mathematical limit to how large the values can become. The minimum value for each variable ($x = 80, y = 130, z = 30$, and $a = 20$) occurs when $w = 0$, because any other value leads to a negative number of cars per hour for some variable, which makes no sense.

The equations still form a chain in which you may choose one value and find the remaining values one step at a time. However, there are now five equations and five variables. In this case, the fifth equation is not needed, because it is automatically satisfied when the other four are.

Proving that the fifth equation is automatically satisfied

To prove that any fifth equation is automatically satisfied, you may again use substitution or elimination. The process requires one more step than in Problem #1, because there is one more equation. For example:

Begin with the South (entering) equation.

$$y - a = 110$$

Use the East equation to obtain an expression for y.

$$y = 100 + z$$

Substitute this expression for y into the South (entering) equation.

$$100 + z - a = 110$$

Use the North equation to obtain an expression for z.

$$z = 30 + w$$

Substitute this expression for z into the previous equation.

$$100 + 30 + w - a = 110$$

Use the West equation to obtain an expression for w.

$$w = x - 80$$

Substitute this expression for z into the previous equation.

$$100 + 30 + x - 80 - a = 110$$

Rearrange and simplify to obtain the equation for South (exiting).

$$130 + x - 80 - a = 110$$
$$50 + x - a = 110$$
$$x - a = 60$$

Students could use the elimination (addition) method as well.

Add the opposite of the East equation to the South (entering) equation.

South (entering)	$y - a = 110$
East	$-y + z = -100$
Sum	$z - a = 10$

Add the North equation to the opposite of the Sum equation.

Sum	$-z + a = -10$
North	$z - w = 30$
New Sum	$a - w = 20$

Add the West equation to the opposite of the New Sum equation.

New Sum	$-a + w = -20$
West	$x - w = 80$
Final Sum	$x - a = 60$

The Final Sum is the South (exiting) equation!

As before, students may use either substitution or elimination with *any* set of four equations to produce the fifth.

The number of variables (5) is again 1 greater than the number of *independent* equations (4), resulting in 1 *free* variable that may take on a range of values. Consequently, there is an infinite number of solutions.

The significance of the infinite number of solutions for traffic flow

The infinite number of solutions suggests that there will again be many traffic configurations consistent with smooth traffic flow. However, if you take a closer look at the specific routes that drivers might choose, you may reach a different conclusion.

Imagining the specific routes that drivers might take

- Enter from the south. Drive along the y region. Exit to the east.
- Enter from the south. Drive along the y and z regions. Exit to the north.
- Enter from the south. Drive along the y, z, w, and x regions. Exit to the south.
- Enter from the west. Drive along the x region. Exit to the south.
- Enter from the west. Drive along the x and y regions. Exit to the east.
- Enter from the west. Drive along the x, y, and z regions. Exit to the north.

The third and fourth options in this list are new, because they involve exiting to the south, which was not possible before. It is interesting that $w = 0$ is still part of a possible solution even though w appears in one of the routes. However, drivers will use this part of the roundabout only if they (1) make a "U-turn" from the south or (2) miss their turnoff. Both of these options are likely to be somewhat rare.

The value of a is even more interesting. It does not appear in any of the routes, because drivers would use this (small) part of the roundabout only if they miss their turnoff. However, the minimum value for a consistent with smooth traffic flow is 20! Perhaps the roundabout illustrated in this problem needs to be redesigned in order to avoid possible traffic congestion issues.

Describing the infinite number of solutions

You may write the solution to this problem as:

$$(w, w+80, w+130, w+30, w+20);\ w \text{ is a whole number}$$

where the ordered quintuple represents (w, x, y, z, a).

If you choose x as the free variable, you will get:

$$(x-80, x\ , x+50, x-50, x-60);\ x \text{ is an integer greater than or equal to 80.}$$

You may also choose y, z, or a as the free variable.

Comparing Problems #1 and #2

Earlier sections in the Solutions describe many of the comparisons. In summary:

- In the scenario from Problem #2, the overall amount of traffic entering and leaving the roundabout is greater than in Problem #1. This might be an important consideration in the design of the roundabout. For example, its radius may need to be greater or more lanes may be required to accommodate a larger number of vehicles.
- Both problem situations lead to a chain of two-variable equations in which one equation is dependent on the remaining ones.
- In both problems, you can use substitution or elimination to prove that one equation is a consequence of (dependent on) the others.
- Both sets of equations have one free variable and an infinite number of solutions that show a clear pattern.

- You may solve both systems of equations quickly by guessing and testing numbers.
- While the new exit from the roundabout in Problem #2 leads to an additional equation, the existing values of the variables in the solutions do not change except for the presence of a new variable having its own values.
- In both problems, it is fairly easy to analyze possible routes that drivers might take. And in both cases, doing so gives additional insight into the likely traffic patterns. In Problem #2, the analysis is especially useful, because it suggests that the solutions for smooth traffic flow may not happen in reality!

What it might mean to exclude the variable a

Excluding a from the model may make sense, especially if the two one-way streets in the south are actually a single two-way street. The system of equations (and thus the solutions for smooth traffic flow) would be the same as in Problem #1. However:

(1) The total amount of traffic entering and exiting the roundabout is still greater than in Problem #1, and

(2) The fact that drivers have the option of exiting the roundabout traveling south creates more possibilities for the routes.

The overall volume of traffic in the roundabout will be greater due both to the increased number of vehicles and the fact that some cars may spend more time in the roundabout (such as when they reverse direction by entering and exiting at the south end). In particular, it is likely that the w-section will be used a little more often.

Stage 2

In Problem #3, students analyze traffic flow in a set of typical four-way intersections and compare their results to the roundabout situations. Although the general mathematical processes are the same, the structure of the equations leads to new features in the solutions. In particular, the values of the variables have both lower *and upper* bounds, and there is a finite number of solutions.

Problem #4 is purely mathematical. Students explore processes for solving three-equation linear systems that look different than the "chains of equations" that have appeared in the traffic problems. Because the problems are not tied to a real-world situation, there is no need to restrict solutions to whole number values.

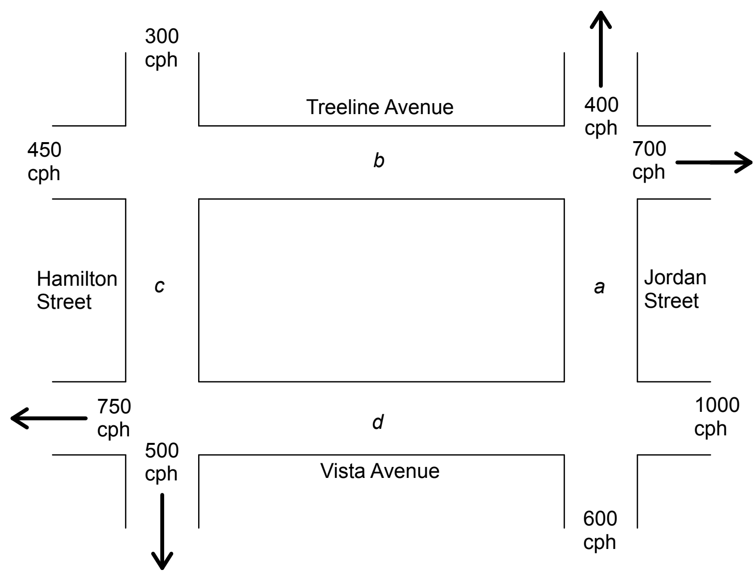
What students should know

- Use the substitution or elimination algorithms to solve systems of two linear equations in two variables.

What students will learn

- Recognize systems of linear equations that have multiple solutions.
- Make connections between numeric and algebraic methods of solving equations.
- Examine patterns in solutions to systems of linear equations.
- Find maximum and minimum values for solutions in real-world contexts.
- Describe solutions efficiently when there is a large number of them.
- Recognize dependent sets of equations (and prove their dependence).
- Create, analyze, and interpret mathematical models.
- Develop methods for solving systems of three linear equations.

Problem #3



Directions

- Figure out as much as you can about the traffic flow in this set of intersections.
- Compare and contrast the situation and your answers with Problems #1 and #2.

Solutions for #3

Note: Comparisons to Problems #1 and #2 are integrated throughout these Solutions.

The hourly flow in and out of the entire network

The flow in and out of the traffic network is already balanced (at 2350 cars per hour) as shown as the following "$\text{in} = \text{out}$" equation. The total volume of traffic is higher than in the Stage 1 problems, and there is no "?" value to find in this case!

$$300 + 450 + 600 + 1000 = 750 + 500 + 400 + 700$$

The values of $a, b, c,$ and d (using a table)

Students' tables should look something like the one below.

a	*b*	*c*	*d*
1100	0	750	500
1050	50	700	550
1000	100	650	600
950	150	600	650
	⋮		
500	600	150	1100
450	650	100	1150
400	700	50	1200
350	750	0	1250

As before, there are many solutions showing clear patterns. However, unlike Problems #1 and #2, the number of solutions is finite!

Patterns in the table

When one variable changes by a certain amount, the other three change by the same amount, but unlike the Stage 1 problems, when a and c change in one direction, b and d change in the opposite direction! As a result, the values of the variables have lower *and upper* bounds, meaning that there is no longer an infinite number of solutions. (In fact, there are exactly 751 solutions.) The range of values for each variable is summarized below.

$$350 \le a \le 1100$$
$$0 \le b \le 750$$
$$0 \le c \le 750$$
$$500 \le d \le 1250$$

Jordan Street and Vista Avenue are carrying a significantly greater traffic load than Hamilton Street and Treeline Avenue.

The system of equations

Working counterclockwise from the upper-right (UR) intersection and arranging the equations with the variables on the left and numbers on the right gives the system:

Upper right (UR)	$a + b = 1100$
Upper left (UL)	$b + c = 750$
Lower left (LL)	$c + d = 1250$
Lower right (LR)	$a + d = 1600$

Patterns in the equations

The equations again reflect relationships in the table. As in Problems #1 and #2, they form a chain in which each equation contains two variables whose coefficients are 1. However, the different geometry of the roads compared to the roundabout results in the variables being added rather than subtracted, which explains the existence of upper bounds on the variables. For example:

- When you increase a, the UR equation forces b to decrease.
- The decrease in b in the UL equation forces c to increase.
- The increase in c in the LL equation forces d to decrease.

Any one of the equations in the system is dependent upon the other three. In other words, when three of the equations are satisfied by values of a, b, c, and d, the fourth is automatically satisfied. For example, suppose that you choose $a = 700$.

Chosen value	$a = 700$
From the UR equation	$b = 400$
From the UL equation	$c = 350$
From the LL equation	$d = 900$

The LR equation is automatically true!

$$a + d = 700 + 900 = 1600$$

You may use substitution or elimination to prove that this *always* happens. For example, using substitution:

Begin with the UR equation.

$$a + b = 1100$$

Use the UL equation to obtain an expression for b.

$$b = 750 - c$$

Substitute this expression for b into the UR equation.

$$a + 750 - c = 1100$$

Use the LL equation to obtain an expression for c.

$$c = 1250 - d$$

Substitute this expression for c into the previous equation.

$$a + 750 - (1250 - d) = 1100$$

Simplify and rearrange to obtain the LR equation.

$$a + 750 - 1250 + d = 1100$$
$$a - 500 + d = 1100$$
$$a + d = 1600$$

Since any one of the equations is dependent on the other three, the system contains three independent equations with four variables, meaning that one of the variables may be chosen freely. As always, this results in multiple solutions. However, in this case, it does not lead to an infinite number of solutions because

(1) There is now an upper bound on the value of each variable.
(2) As before, only whole number values of the variable make sense.

Describing the solutions

If you choose b as the free variable, the solution to this problem looks like

$(1100 - b, b, 750 - b, 500 + b)$; b is a whole number less than or equal to 750.

The ordered quadruple represents (a, b, c, d).

If you choose a as the free variable, you will get

$$(a, 1100 - a\ , a - 350, 1600 - a);$$

a is a whole number between 350 and 1100 inclusive.

You may also choose c or d as the free variable.

Managing traffic flow during construction

Closing the blocks on Treeline Avenue and Jordan Street (one at a time) may not cause much trouble, because solutions for smooth traffic flow exist when there is no traffic on these parts of the roads. However, individual drivers will have to adjust their routes, which may lead to unexpected consequences unless you analyze the routes that drivers are likely to take. (See the discussions of individual routes in Stage 1.)

Closing Hamilton Street or Vista Avenue is likely to result in greater disruption to traffic because (1) these roads carry more traffic, and (2) there is no solution for smooth traffic flow consistent with having 0 cars per hours on these roads. In order to manage to these situations, it may be necessary to take steps such as (1) closing only one lane of traffic at a time (if the roads have more than one lane), (2) using detours to divert traffic to other roads, and/or (3) adjusting traffic signals or signs to control traffic flow.

Imagining the specific routes that drivers might take

While it may be valuable to analyze individual routes in this case, there are many more of them to consider and the analysis might be complex. Given that there is an entrance and an exit at every intersection, the geometry of the situation seems more symmetrical than it was with the roundabouts, and perhaps this type of analysis will be less helpful. On the other hand, the amount of traffic entering and leaving each intersection is not very symmetrical, which may make the analysis more worthwhile. Some students may be interested in exploring these ideas!

The value of knowing about traffic flow on different roads

If you are preparing to build an intersection, anticipating the traffic flow on the roads leading in and out of it will have an impact on how you design the intersection. The size, number of lanes, traffic signals and signs, etc. may depend on this information. If the roads and intersections are already built, the knowledge may help to predict wear and tear, anticipate changes that will need to be made in the future, evaluate safety issues, etc.

Problem #4

System 1

$$\begin{aligned} a - b + c &= 19 \\ -a - b - c &= -23 \\ a - 3b + c &= 15 \end{aligned}$$

System 2

$$\begin{aligned} 3a + b + c &= 7 \\ a + 2b - 4c &= 14 \\ -2a - b + c &= -3 \end{aligned}$$

Directions

- Solve each system using methods of your choice. Explain your thinking.
- Describe the important features of each system, such as whether the equations are dependent or independent, the number of free variables, the number of solutions, etc.

Diving Deeper

- Can you create traffic scenarios that correspond to these systems? If so, show how to do it. If not, explain what makes it difficult or impossible.

Solutions for #4

Solving System 1

Using ordered triples to represent (a, b, c), the solution is

$$(a, 2, 21 - a).$$

In other words, a may be any number, b is always equal to 2, and c is equal to 21 minus the value you choose for a.

You may also write the solutions as $(21 - c, 2, c)$.

Many combinations of strategies are possible.

Some students may begin by guessing values. For example, suppose they start with a. Regardless of the number they choose, when they substitute it for a in each of the three equations, it always produces a solution. The value of b in every solution will be 2, and the value of c will always be equal to 21 minus the value that they chose for a. (If they guess a value for c, the situation will be reversed. b will still equal 2, but the value for a will be 21 minus the value that they chose for c.)

If they attempt to guess values for b, none of them will work— unless they happen to guess the number 2. If they do choose 2 and they substitute it into all three equations, every equation will be equivalent to $a + c = 21$! Because this equation provides the only available relationship between a and c, it is impossible to pin down a specific value for either number.

Some students may recognize patterns in the system of equations enabling them to predict that $b = 2$ without guessing. For example, the first and last terms of Equation 1 and Equation 3 are the same, but the middle term "decreases" by $2b$. Since the number on the right decreases by 4, it must be that $-2b = -4$ and therefore that $b = 2$.

Students who choose traditional strategies such as substitution or elimination (especially elimination) may find solutions fairly quickly. For example, by adding the first two equations, they obtain the same equation as in the preceding paragraph: $-2b = -4$.

Solving System 2

System 2 has only one solution: $(-6, 20, 5)$. That is, $a = -6$, $b = 20$, and $c = 5$.

Because there is only one solution, students who guess values for a, b, or c may have less success than in the past. If they are lucky enough to choose one of the values above, they may substitute it into each equation (or just two of them) and produce a system of two equations and two variables that they know how to solve. However, if they do not stumble onto one of the correct values for a, b, or c, they may hit a roadblock.

In this case, students may need to find ways to apply their knowledge of substitution and/or elimination to this larger system. If they try substitution and struggle with it, suggest that they try elimination. A common approach is to eliminate the same variable from two pairs of equations. For example, eliminate the variable b from Equations 1 and 3 and from Equations 2 and 3.

Equation 1 + Equation 3: $a + 2c = 4$

Equation 2 + 2(Equation 3): $-3a - 2c = 8$

Now that you have two equations in two variables, you may solve them in the usual way to find values for a and c. For example, adding the two equations eliminates c, giving

$$\begin{aligned} -2a &= 12 \\ a &= -6 \end{aligned}$$

Once you know one of the values, everything gets easier! Substituting this value into either of the two equations above gives $c = 5$.

$$\begin{aligned} -6 + 2c &= 4 \\ 2c &= 10 \\ c &= 5 \end{aligned}$$

Finally, substituting these values for a and c into any of the original equations shows that $b = 20$. If you substitute them into Equation 1:

$$\begin{aligned} 3(-6) + b + 5 &= 7 \\ -18 + b + 5 &= 7 \\ -18 + b &= 2 \\ b &= 20 \end{aligned}$$

It is a good idea to test these three values in the other two equations as well.

Note: Since students have not learned detailed methods for handling systems of three equations in three variables, they are likely to need plenty of time to experiment and to try things—many of which will not work well at first. In fact, the point of the problem is to spur creative thinking and to give students experience applying previously learned techniques (such as substitution and elimination) to a new situation.

Important features of the systems

System 1 has an infinite number of solutions. There is one free variable, either a or c. Interestingly, b cannot be the free variable, because it has only one possible value!

Because there are three variables and one of them is free, $3 - 1 = 2$ of the equations are independent, and the third one must be dependent on the other two. Students may use either substitution or elimination to prove this, but elimination may be easier. For example, they may show that Equation 3 is dependent on Equations 1 and 2 by adding twice the first equation to the second one.

2E1	$2a - 2b + 2c = 38$
E2	$-a - b - c = -23$
2E1 + E2	$a - 3b + c = 15$

The final equation is the same as Equation 3! Since this equation is an automatic consequence of the first two, it is dependent on them.

Students may show any one of the three equations to be dependent on the other two, and they may use either substitution or elimination (or a combination of them) to prove it.

System 2 has only one solution. There are no free variables. That is, no variable may take on more than one value. As a result, the number of independent equations is equal to the number of variables, 3. In other words, all three equations are independent. None of them may be derived from the other two.

Notes: Many real-world situations require systems of hundreds, thousands, or more equations and variables! The solutions to these systems will often not be restricted whole numbers. In high school, students will learn techniques involving *matrices* for managing these types of calculations. In practice, the calculations are typically performed by computers.

Stage 3

Problem #4 offered students some experience with systems of equations that were harder to solve by guessing and testing numbers and looking for patterns (because the systems did not consist of chains of equations as they did in the first few problems). In Stage 3, students will apply their knowledge from this problem and earlier ones to analyze a more complex traffic situation. The picture looks a lot the same as in Problem #3, but a new road has been added and the variables occur in different places.

Although the learning goals listed below are the same as in Stages 1 and 2, the ideas and procedures are more complex and challenging to apply in Problem #5. There are more equations and more variables, and the solution to the system involves *two* free variables.

Some students may be interested in reading about methods for using *matrices* to solve large systems of linear equations. If they search for traffic problems and linear systems on the internet, they will find many videos that discuss these methods, including the *Gauss-Jordan elimination* algorithm. Ironically, although the makers of these videos often apply the Gauss-Jordan technique to solve their problems, the numeric approach described in this activity is probably easier and more practical for the types of systems (with chains of equations) that often arise when studying traffic flow.

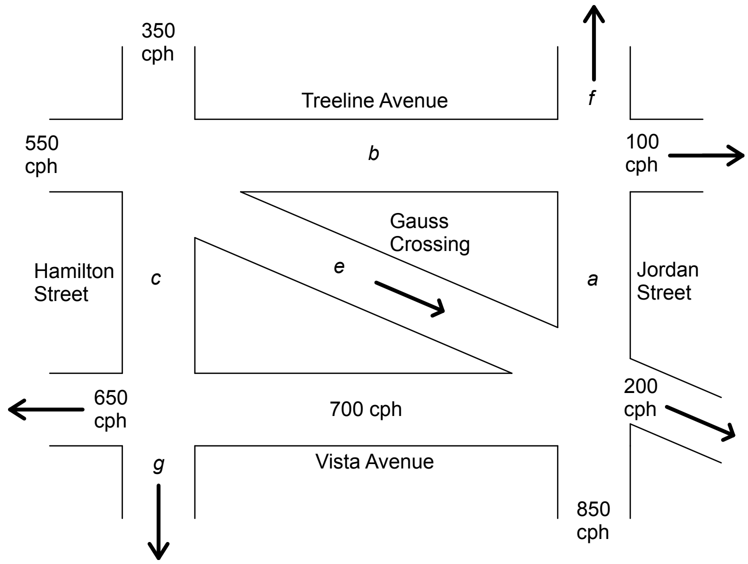
What students should know

- Use the substitution or elimination algorithms to solve systems of two linear equations in two variables.

What students will learn

- Recognize systems of linear equations that have multiple solutions.
- Make connections between numeric and algebraic methods of solving equations.
- Examine patterns in solutions to systems of linear equations.
- Find maximum and minimum values for solutions in real-world contexts.
- Describe solutions efficiently when there is a large number of them.
- Recognize dependent sets of equations (and prove their dependence).
- Create, analyze, and interpret mathematical models.

Problem #5



Directions

- Find and describe all possible sets of values for the variables that might lead to smooth traffic flow. Include upper and lower bounds for each variable.
- Analyze the important features of the system of equations, such as the dependence or independence of the equations, the number of free variables, etc.

Diving Deeper

- How many solutions does this problem have?

Solutions for #5

The solution (written with algebraic expressions)

The solution written as ordered 6-tuples of the form (a, b, c, e, f, g) is

$$(a,\ 850 - a - c,\ c,\ a + 50,\ 750 - c,\ c + 50).$$

In other words:

a is a free variable	$b = 850 - a - c$	c is a free variable
$e = a + 50$	$f = 750 - c$	$g = c + 50$

If you use variables other than a and c as the free variables, the solution *looks* different, but it represents the same set of 6-tuples.

The problem situation restricts the values of the variables.

$0 \leq a \leq 850 - c$	$0 \leq b \leq 850 - c$	$0 \leq c \leq 750$
$50 \leq e \leq 900 - c$	$0 \leq f \leq 750$	$50 \leq g \leq 800$

This description results from (1) choosing c and a as the free variables, and (2) choosing the value for c first. Other choices result in different representations for the restrictions.

Students may use a combination of numeric and algebraic methods to discover the solutions and restrictions. In any case, it helps to begin with the system of equations.

The system of equations

$$\begin{array}{lrcl}
(1) & f + g & = & 800 \\
(2) & b + c + e & = & 900 \\
(3) & -c + g & = & 50 \\
(4) & -a + e & = & 50 \\
(5) & a + b - f & = & 100
\end{array}$$

The first equation expresses the condition that the total number of cars entering the diagram is equal to the number of cars exiting. The remaining equations follow from the same condition applied to the individual intersections beginning in the upper left and traveling clockwise around the diagram.

Creating a table

Because the number of equations (5) is one less than the number of variables (6), there will be at least one free variable. (There will be more than one if any of the five equations are dependent on the others.)

By choosing a value for c, you may find corresponding values for g and f quickly. For example, suppose that $c = 200$. Then $-c + g = 50$ implies that $g = 250$, which in turn implies (using $f + g = 800$) that $f = 550$.

Once you determine these values, you can substitute them into the remaining formulas to obtain a chain three two-variable equations with three unknowns. In this case, the equations are

$$b + e = 700$$
$$e - a = 50$$
$$a + b = 650$$

Adding the second two equations produces the first one. Therefore, you may choose another variable freely. If you use the variable a, you get many sets of triples that correspond with $c = 200$, $f = 550$, and $g = 250$. For example, if you choose $a = 0$, you obtain $b = 650$ and $e = 50$. By continuing to choose values for c and a, you can gradually build tables of solutions. (See the next page.)

The process of writing the solution with variables

By analyzing their tables closely, students may write solutions using variables.

- List a and c as free variables, because students chose their values.

Work to write all remaining variables in terms of a and c.

- Recognize that e is 50 greater than a in every solution. Write: $e = a + 50$.
- Recognize that g is 50 greater than c in every solution. Write: $g = c + 50$.
- Recognize that c and f have a sum of 750 in every solution.
  Write: $f = 750 - c$.
- Notice that the value of b depends on both a and c. Specifically, the sum of a, b, and c is always 850. Write: $b = 850 - a - c$.

Students may also use the original equations directly.

- They can find expressions for e and g directly from equations 3 and 4.
- They can find an expression for f by combining equations 1 and 3. Substitute $g = c + 50$ into $f + g = 800$ in order to obtain $f + c + 50 = 800$, which is equivalent to $f + c = 750$ (or simply subtract the two equations).
- They can find an equation for b by combining equations 2 and 4. Substitute $e = a + 50$ into $b + c + e = 900$ in order to obtain $b + c + a + 50 = 900$, which is equivalent to $b + c + a = 850$ or $b = 850 - a - c$.

Notice that equation 5 was not used in any of these calculations, which may lead you to suspect that equation 5 is not a necessary part of the system! In other words, it is dependent on the other four equations. See the next section of the Solutions to explore this idea further.

Tables of Selected Solutions for Problem #5

c*, *f*, *g	**Equations**		***a*, *b*, *e***		
$c = 0$	$b + e = 900$	$a = 0$	$a = 50$		$a = 850$
$f = 750$	$e - a = 50$	$b = 850$	$b = 800$	$\cdots$	$b = 0$
$g = 50$	$a + b = 850$	$e = 50$	$e = 100$		$e = 900$

$c = 50$	$b + e = 850$	$a = 0$	$a = 50$		$a = 800$
$f = 700$	$e - a = 50$	$b = 800$	$b = 750$	$\cdots$	$b = 0$
$g = 100$	$a + b = 800$	$e = 50$	$e = 100$		$e = 850$

$c = 100$	$b + e = 800$	$a = 0$	$a = 50$		$a = 750$
$f = 650$	$e - a = 50$	$b = 750$	$b = 700$	$\cdots$	$b = 0$
$g = 150$	$a + b = 750$	$e = 50$	$e = 100$		$e = 800$

$\vdots$

$c = 650$	$b + e = 250$	$a = 0$	$a = 50$		$a = 200$
$f = 100$	$e - a = 50$	$b = 200$	$b = 150$	$\cdots$	$b = 0$
$g = 700$	$a + b = 200$	$e = 50$	$e = 100$		$e = 250$

$c = 700$	$b + e = 200$	$a = 0$	$a = 50$		$a = 150$
$f = 50$	$e - a = 50$	$b = 150$	$b = 100$	$\cdots$	$b = 0$
$g = 750$	$a + b = 150$	$e = 50$	$e = 100$		$e = 200$

$c = 750$	$b + e = 150$	$a = 0$	$a = 50$	$a = 100$	
$f = 0$	$e - a = 50$	$b = 100$	$b = 50$	$b = 0$	
$g = 800$	$a + b = 100$	$e = 50$	$e = 100$	$e = 150$	

Free variables and the number of independent equations

The system of equations appears to be dependent for at least two reasons.

- Only 4 of the 5 equations are needed to write the solutions algebraically.
- Since the solution contains 2 free variables, the total number of variables should be 2 greater than the number of independent equations. Because there are 6 variables, only 6 – 2 = 4 of the equations are independent. One of the five equations must be dependent on the others.

To prove this directly, use four of the equations to derive the fifth. For example, use equations (2) through (5) to derive equation (1).

Begin with equation (5):

$$a + b - f = 100$$

Use equation (4) to substitute $e - 50$ for a.

$$e - 50 + b - f = 100$$
$$e + b - f = 150$$

Use equation (2) to substitute $900 - c$ for $e + b$.

$$900 - c - f = 150$$
$$c + f = 750$$

Use equation (3) to substitute $g - 50$ for c.

$$g - 50 + f = 750$$
$$f + g = 800$$

The final result is equation (1), which proves that it is dependent on equations (2) through (5). In other words, it is automatically true whenever they are.

Notice that *all four* equations (2) through (5) play a role in deriving equation (1)! If any of them are missing, we cannot derive the final equation. Also, though the process may look random, there is a clear goal: eliminate every variable except the f and g that appear equation (1). Since equation (5) already contains f, the idea is to transform the a and b from this equation into g, one step at a time. Beginning with f, a, and b:

- Equation (4) transforms a into e, leaving f, e and b.
- Equation (2) transforms e and b into c, leaving f and c.
- Equation (3) transforms c into g, leaving f and g.

If it had not been possible to create an equation containing only f and g, then equation (1) would not have been dependent on the others. If it *had* been possible to create such an equation, but the sum of f and g came out to be something other than 800, the system would have had no solution!

Finding the restrictions on the values of the variables

Each variable must be a whole number, because it represents a number of cars per hour. Ensuring that *every* variable is a whole number places further restrictions on all of their values.

The table and equations contain the information needed to predict and analyze these restrictions. The analysis might be fairly complex at times. A couple of examples:

The variable a

The least value of a that appears in any solution is 0.

The greatest value of a that appears in any solution is 850, and this occurs when $c = 0$. As you increase the value of c, the maximum value of a decreases by c.

If a were greater than $850 - c$:

$a > 850 - c$	Assumption
$-a < c - 850$	Multiply both sides by –1.
$-a + e < c - 850 + e$	Add e to both sides.
$50 < c - 850 + e$	From equation (4), $-a + e = 50$.
$c + e > 900$	Add 850 to both sides, and write the inequality in reverse.

Equation (2) would then force b to be negative.

Conclusion: $0 \leq a \leq 850 - c$.

The variable e

The least value of e appearing in any solution is 50.

The greatest value of e that appears in any solution is 900, and this occurs when $c = 0$. As you increase the value of c, the maximum value of e decreases by c.

If e were less than 50, then equation (4) would force a to be negative.

If e were greater than $900 - c$:

$e > 900 - c$	Assumption
$b + c + e > 900 - c + b + c$	Add $b + c$ to both sides.
$b + c + e > 900 + b$	Simplify. $(-c + c = 0)$
$c + e > 900$	Subtract b from both sides.

Again, equation (2) would force b to be negative.

Conclusion: $50 \leq e \leq 900 - c$.