Deep Algebra Projects: Pre-Algebra/Algebra 1
Decimal Patterns

Topics

- Fraction ⇔ decimal conversions that involve repeating decimals
- Patterns in repeating decimals
- Definitions of *rational numbers*

In the *Decimal Patterns* project, students delve deeply into patterns that arise when translating between fraction and decimal forms of a number. In the process, they begin to make sense of the two equivalent definitions of a *rational number*.

The focus throughout the project is on having students create their own strategies. This approach to learning helps students develop problem solving skills, number and operation sense, and intuition about why different methods make sense.

Some readers may have seen an algebraic approach for converting repeating decimals into fraction form. This approach is not mentioned in the solutions, because it is unlikely to be discovered independently by students, and it is based on concepts that they may not yet have studied in enough depth to make complete sense of the approach. In case you are curious and/or you believe that the approach would be beneficial to your students, I am including two examples of it below. If you do choose to share it with your students, I would recommend waiting until they have first completed as many of the problems in this project as they are able using their own methods.

A traditional algebraic approach to writing repeating decimals in fraction form:

$$\begin{aligned} n &= 0.\overline{79} \\ 100n &= 79.\overline{79} \\ -n &\quad -0.\overline{79} \\ \hline 99n &= 79 \\ n &= \frac{79}{99} \end{aligned}$$

$$\begin{aligned} n &= 0.4\overline{21} \\ 1000n &= 421.\overline{21} \\ -10n &\quad -4.\overline{21} \\ \hline 990n &= 417 \\ n &= \frac{417}{990} = \frac{139}{330} \end{aligned}$$

Consider showing these to students without explaining them. See how much progress they can make in understanding and trying the methods on their own.

Stage 1

In Problem #1, students begin developing strategies to write repeating decimals in fraction form. They compare familiar decimals with unfamiliar ones and use their observations to find fractions for the unfamiliar ones. This process encourages students to think of decimals and fractions as *numbers* as well as numerals—that is, as quantities with a *size* as well as symbols on a page. In particular, they see that when the size of a number changes, its different representations change correspondingly. For example, when a fraction decreases by $\frac{1}{10}$, the corresponding decimal does the same.

Problems #2 and #3 introduce a division algorithm that may not be familiar to students. I have chosen the algorithm because I believe it makes patterns a little easier to see. If you prefer to have your students stick with a familiar algorithm, it should not cause any difficulties. In either case, students will calculate decimal expressions for simple fractions, analyze patterns, and apply the patterns to make predictions about other decimals.

What students should know

- Be fluent with at least one division algorithm for whole numbers and decimals.
- Know decimal expressions for halves, thirds, fourths, fifths, and tenths.
- Translate between fraction and decimal form for terminating decimals.

What students will learn

- Understand fractions and decimals more deeply as alternate representations of numbers.
- Begin to develop strategies to find fraction expressions for repeating decimals.
- Analyze patterns in decimals, and extend them to find previously unknown decimal expressions for fractions.
- Begin to explore *periods* of repeating decimals.

Problem #1

$$\frac{1}{4} = 0.25 \qquad\qquad \frac{1}{3} = 0.\overline{3}$$

$$\frac{1}{8} \qquad\qquad \frac{1}{9}$$

Directions

- Use the given decimals to find decimal expressions for $\frac{1}{8}$ and $\frac{1}{9}$. Explain your thinking.
- Use the decimal expressions for $\frac{1}{8}$ and $\frac{1}{9}$ to find decimal expressions for other *8ths* and *9ths* fractions. Explain your thinking.

Diving Deeper

When a fraction is written in simplest from, the prime factorization of the denominator will tell you whether or not the fraction's decimal expression will *terminate* (end). Explain how and why.

Conversation Starters for #1

What do you notice? What do you wonder?

I notice that the first decimal terminates (ends) and the second one repeats.

I notice that the decimals are missing for the fractions on the bottom.

I notice that all of the numerators are 1.

I notice that the first denominator doubles and the second one triples.

I notice that first fraction is half the one above it and the second fraction is a third of the one above it.

I wonder if I can use the relationships between the fractions in order to find the missing decimal expressions.

I wonder how I can use the value of the decimal for the *unit* fractions to find decimal expressions for other fractions with the same denominator.

I wonder if there will be patterns in the decimals.

I notice that, according to the *9ths* pattern, the decimal for $\frac{9}{9}$ is $0.\overline{9}$.

I wonder why the *9ths* pattern gives an answer of $0.\overline{9}$ for $\frac{9}{9}$ when it should be 1.

This is a great opportunity to talk about the (true) equation, $0.\overline{9} = 1$. Don't worry if your students are skeptical! It is very reasonable to doubt this. They may argue that $0.\overline{9}$ never "gets to" 1. You might point out that it is equally true that $0.\overline{3}$ never "gets to" $\frac{1}{3}$. However, they may find the equation $0.\overline{3} = \frac{1}{3}$ easier to believe, because they can produce the decimal using a division algorithm on 1 ÷ 3. *Note*: I am not sure how you might get $0.\overline{9}$ by using a division algorithm on 9 ÷ 9. Perhaps students can think of a way!

Solutions for #1

Finding decimal expressions for $\frac{1}{8}$ and $\frac{1}{9}$

$$\frac{1}{8} = 0.125 \qquad \frac{1}{9} = 0.\overline{1}$$

Since $\frac{1}{8}$ is half of $\frac{1}{4}$, you can find the decimal for $\frac{1}{8}$ by taking half of 0.25. Some students may think in terms of percentages: half of 25% is 12.5%, which is 0.125. Others may think directly in terms decimals: 0.25 = 0.250. Half of 0.250 is 0.125. (The decimals may make more sense if you think of the place values. Number-word notation is good for this: 25 hundredths = 250 thousandths. Half of 250 thousandths is 125 thousandths. Students could also think of the numbers as fractions.)

Since $\frac{1}{9}$ is a third of $\frac{1}{3}$, the decimal for $\frac{1}{9}$ is one third of $0.\overline{3}$, which is $0.\overline{1}$.

Decimal expressions for the eighths and ninths

$$\frac{1}{8} = 0.125 \quad \frac{2}{8} = 0.25 \quad \frac{3}{8} = 0.375 \quad \frac{4}{8} = 0.5$$
$$\frac{5}{8} = 0.625 \quad \frac{6}{8} = 0.75 \quad \frac{7}{8} = 0.875 \quad \frac{8}{8} = 1$$

$$\frac{1}{9} = 0.\overline{1} \quad \frac{2}{9} = 0.\overline{2} \quad \frac{3}{9} = 0.\overline{3} \quad \frac{4}{9} = 0.\overline{4} \quad \frac{5}{9} = 0.\overline{5}$$
$$\frac{6}{9} = 0.\overline{6} \quad \frac{7}{9} = 0.\overline{7} \quad \frac{8}{9} = 0.\overline{8} \quad \frac{9}{9} = 0.\overline{9} = 1$$

Students may find many methods. A common approach is to keep adding 0.125 for the eighths and $0.\overline{1}$ for the ninths. For many of the *eighths* fractions, they can use memorized values of the simplified forms. They may also build off of these fractions to find other values. For example, to find the decimal for $\frac{7}{8}$, they may start with $\frac{3}{4}$ (0.75) and add $\frac{1}{8}$ (0.125). Or they may start with 1 and subtract 0.125.

Students are likely to observe that the *ninths* fractions are particularly easy to handle, because the numerator of the fraction is the repeating digit in the decimal. Their results for $\frac{9}{9}$ should lead them to the (correct) conclusion that $0.\overline{9} = 1$, which may lead to some interesting discussions! (See the Conversation Starters.)

Problem #2

$$\begin{array}{r} 0.1\;6\;6\;6 \\ 6\overline{)1.{}^{1}0\,{}^{4}0\,{}^{4}0\,{}^{4}0} \end{array}$$

Directions

- Figure out and explain how and why this division algorithm works.
- Use the algorithm to find a decimal expression for $\frac{5}{6}$.
- If you have any observations or see any interesting patterns, describe them.
- Add the decimal expressions for $\frac{1}{6}$ and $\frac{5}{6}$. Explain why the answer makes sense.

Diving Deeper

- Use the division algorithm above to find a decimal expression for $\frac{1}{11}$.
- Use pencil/paper or mental math methods of your choice to find decimal expressions for other "elevenths" fractions. Explain your thinking and describe any patterns that you see.

Conversation Starters for #2

What do you notice? What do you wonder?

I notice that there are no subtraction processes written out below the dividend (unlike the usual division algorithm).

I wonder what the small numbers above and to the left of the 0s in the dividend are all about.

I wonder if would help to do the division my usual way and then compare it to the algorithm on the problem page.

I notice that the numbers above and to the left of the 0s are the same as the remainder that I get when I divide my usual way.

I notice that ${}^{1}0$ reminds me of 10 and ${}^{4}0$ reminds me of 40.

I notice that the subtraction calculations for 1 ÷ 6 are easy to do mentally.

I notice that the division process will continue forever.

I notice that I can find the decimals for all of the other *6ths* fractions, because if I simplify them, I get fractions whose decimals I already know.

I notice that the repeating digits for the *6ths* fractions (3 and 6) are the same as ones in the *3rds* fractions.

I wonder if it is a coincidence that 3 is a factor of 6, and the repeating digits for the *3rds* and *6ths* fractions are the same.

It is not a coincidence. Students will have a chance to explore this idea further, especially if they work on Problem #8.

I notice that the decimal $0.\overline{9}$ shows up again in the last question—and again it looks like it should equal 1!

Solutions for #2

How the algorithm works

Write the dividend as 1.0.

$$1 \div 6 \rightarrow 0 \text{ r } 1$$

Write 0 above the 1, and write the remainder, 1, before the next 0-digit.

$$\begin{array}{r} 0. \phantom{^{1}0} \\ 6\overline{)1.^{1}0} \end{array}$$

Read $^{1}0$ as "10."

$$10 \div 6 \rightarrow 1 \text{ r } 4$$

Write the quotient as the digit 1 above the $^{1}0$.

Write the remainder, 4, before a new digit, 0, in the hundredths place.

$$\begin{array}{r} 0\ .1 \phantom{^{4}0} \\ 6\overline{)1.^{1}0^{4}0} \end{array}$$

Read $^{4}0$ as "40."

$$40 \div 6 \rightarrow 6 \text{ r } 4$$

Write the quotient as the digit "6" above the $^{4}0$.

Write the remainder, 4, before a new digit, 0, in the thousandths place.

$$\begin{array}{r} 0\ .1\ \ 6 \phantom{^{4}0} \\ 6\overline{)1.^{1}0^{4}0^{4}0} \end{array}$$

Continue the process. Notice that it will go on forever, because the remainder will always be 4. The quotient will read 0.1666666… or $0.1\overline{6}$.

Why the algorithm works

The best way for students to understand why the algorithm works is the write out their usual algorithm alongside it. They should observe that key features are the same, but (1) the work has been collapsed vertically, (2) the subtraction is done mentally. Those who use the traditional algorithm may notice that the process of writing the remainder before the next 0-digit in the dividend has the same effect as "dropping down the 0."

Using the algorithm to find a decimal expression for $\frac{5}{6}$

$$\begin{array}{r} 0\ .8\ \ 3\ \ 3\ \ 3 \phantom{^{2}0} \\ 6\overline{)5.^{5}0^{2}0^{2}0^{2}0} \end{array}$$

Therefore, $\frac{5}{6} = 0.8\overline{3}$.

Observations and patterns

Students may notice many things, including:

- There is one digit at the beginning of each decimal that does not repeat.
- The decimal for $\frac{1}{6}$ has the same digits as the fraction. (This is a coincidence, but it does make the decimal easier to remember.)
- The repeating digits, 6 and 3, are the same ones that appear in the repeating decimals forms for $\frac{2}{3}$ and $\frac{1}{3}$. (This is *not* a coincidence!)
- The digits in corresponding place values in $\frac{5}{6}$ and $\frac{1}{6}$ always have a sum of 9.

The sum of the decimals

0.166666... + 0.833333... = 0.999999...

This makes sense, because $\frac{1}{6} + \frac{5}{6} = 1$, and (as we saw in the previous problem) $0.\overline{9} = 1$.

Some students may discover that this suggests a shortcut for finding a decimal expression for 1 – *n* when you already know the decimal for *n*.

Problem #3

$$\begin{array}{r} 0.1\;4\;2\;8\;5\;7 \\ 7\overline{\big)1.^{1}0^{3}0^{2}0^{6}0^{4}0^{5}0} \end{array}$$

Directions

- Explain what will happen if you continue the algorithm above.
- Use the algorithm to find decimal expressions for $\frac{2}{7}$ and $\frac{3}{7}$.
- Find and describe as many patterns as you can.
- Use your patterns to predict the decimal expressions for the remaining *7ths* fractions. Explain your thinking.
- Explain how you could have predicted (before dividing) that the greatest possible *period** of the repeating decimal is 6.

*****The *period* of a repeating decimal is number of digits in each repeating block.

Conversation Starters for #3

What do you notice? What do you wonder?

I notice that the division processes for all three fractions (with numerators of 1, 2, and 3) seem to involve the same calculations in a different order.

I notice that whenever I get a remainder that I got earlier, the decimal begins to repeat.

I notice that the remainders 1, 2, 3, 4, 5, and 6 all show up in the algorithm.

I notice that the digits in the decimal for $\frac{1}{7}$ break up into 14, 28, and 57. When you double 14, you get 28, and when you double 28, you get 56, which is close to 57.

I wonder if there is a reason for this (almost) doubling.

I notice that the decimals contain the same digits.

I wonder if the same thing happens with the decimals for fractions with other denominators.

I wonder if the decimals for a division problem could ever keep on going forever without repeating.

Solutions for #3

Continuing the algorithm

If you continue the algorithm, the next remainder will be 1 (the same as the first remainder), and the entire process will begin to repeat itself. The quotient will have six repeating digits, $0.\overline{142857}$.

Finding decimal expressions for $\frac{2}{7}$ and $\frac{3}{7}$

$$\begin{array}{r} 0\,.\,2\;\,8\;\,5\;\,7\;\,1\;\,4 \\ 7\overline{)2.^{2}0^{6}0^{4}0^{5}0^{1}0^{3}0} \end{array} \qquad \begin{array}{r} 0\,.\,4\;\,2\;\,8\;\,5\;\,7\;\,1 \\ 7\overline{)3.^{3}0^{2}0^{6}0^{4}0^{5}0^{1}0} \end{array}$$

Therefore, $\frac{2}{7} = 0.\overline{285714}$ and $\frac{3}{7} = 0.\overline{428571}$.

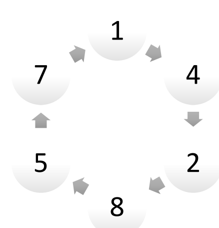
Sample patterns

- There is an "almost doubling" pattern in the decimal for $\frac{1}{7}$: 14, 28, 57.
- You can get the next-greater decimal by adding $0.\overline{142857}$ each time.
- Every decimal has the same six digits. Each digit occurs one time.
- The digits always occur in the same order. When you get to the end of the pattern, they wrap back around to the beginning.
- As the numerator increases, the tenths digits increase.
- The sum of the number formed from the first three digits and the number formed by the last three digits is always equal to 999.

Predicting the decimals for the remaining sevenths *fractions*

Students will have many approaches. For example:

- Add $0.\overline{142857}$ to each decimal in order to get the next-greater one.
- Make pairs that add to add 1: $\frac{1}{7} + \frac{6}{7}$ $\quad \frac{2}{7} + \frac{5}{7}$ $\quad \frac{3}{7} + \frac{4}{7}$. To find the decimal for the greater number in each pair, figure out what to add to the smaller number (decimal) in order to get $0.\overline{999999}$.
- Place the six digits in order: 1, 2, 4, 5, 7, 8. These are the tenths digits of the fractions $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$, respectively. Beginning with each tenths digit, use the pattern below to find the next five digits in each cycle:



Students may divide to check that these strategies give the correct decimals:

$$\frac{4}{7} = 0.\overline{571428} \quad \frac{5}{7} = 0.\overline{714285} \quad \frac{6}{7} = 0.\overline{857142}$$

Predicting the maximum possible period of the decimal

When you divide by 7, there are only seven possible remainders: 0, 1, 2, 3, 4, 5 and 6. If you get a remainder of 0 at any point in the division process, then the decimal will terminate.

If the decimal does not terminate, one of the other six remainders must eventually appear a second time in the division process. At this point, the entire process—and thus the digits in the quotient—must repeat. The maximum possible period occurs if all six remainders show up at some point in the division process (which they do in the case of *7ths*).

Since there is one digit in the quotient for each remainder, the longest possible period is 6.

Note: Based on this reasoning, when you divide by n, the longest possible period of a repeating decimal is $n - 1$. Students may be interested in searching for other divisors for which the decimal attains its maximum possible period.

Stage 2

In Stage 2, students extend their learning from Stage 1 by incorporating ideas about place value, digging into increasingly open-ended investigations, exploring more complex patterns, and analyzing decimals that have both a non-repeating and repeating part. As before, they develop and justify their own strategies.

What students should know

- Understand ideas from Stage 1.
- Add and subtract fractions with unlike denominators.

What students will learn

- Use place value ideas to explore connections between fractions and decimals.
- Investigate new and more complex types of decimal patterns.
- Begin to create and justify strategies to convert decimals with both a non-repeating and repeating part into fraction form.

Problem #4

$$\frac{1}{4} = 0.25 \qquad\qquad \frac{1}{3} = 0.\overline{3}$$

$$\frac{1}{40} \qquad\qquad \frac{1}{300}$$

Directions

- Use the given decimals to find decimal expressions for $\frac{1}{40}$ and $\frac{1}{300}$. Explain your thinking.
- Use a pencil/paper or mental math method(s) of your choice to calculate the decimal expression for $\frac{17}{600}$. Explain the connection to the task above.

Conversation Starters for #4

What do you notice? What do you wonder?

I notice that the first decimal terminates (ends) and the second one repeats.

I notice that the decimals are missing for the fractions on the bottom.

I notice that all of the numerators are 1.

I notice that the first denominator gets 10 times greater, and the second one gets 100 times greater.

I notice that first fraction is one-tenth of the one above it and the second fraction is one-hundredth of the one above it.

I notice that dividing by 10 or 100 shifts the place values in a decimal.

I wonder if I can use the relationships between the fractions in order to find the missing decimal expressions.

I wonder if I can use the value of the decimal for the *unit* fractions to find decimal expressions for other fractions with the same denominator.

I wonder if there will be patterns in the decimals.

I notice that 600 (the denominator of $\frac{17}{600}$) is a multiple of both 40 and 300 (the denominators of the two fractions on the bottom).

I wonder what happens if I add the two fractions on the bottom.

Solutions for #4

Decimal expressions for $\frac{1}{40}$ and $\frac{1}{300}$

$$\frac{1}{40} = 0.025 \qquad \frac{1}{300} = 0.00\overline{3}$$

$\frac{1}{40}$ is one tenth of $\frac{1}{4}$. In order to show this with decimals, the tenths and hundredths digits in 0.25 must each shift one place to the right, leaving a 0 in the tenths place: 0.025.

$\frac{1}{300}$ is one hundredth of $\frac{1}{3}$. In order to show this with decimals, all digits to the right of the decimal point in $0.\overline{3}$ must shift two places to the right, leaving 0s in the tenths and hundredths places: $0.00\overline{3}$.

A decimal expression for $\frac{17}{600}$

$$\frac{17}{600} = 0.028\overline{3}$$

Many students are likely to find this decimal by dividing: 17 ÷ 600.

Those who recognize that $\frac{1}{40} + \frac{1}{300} = \frac{17}{600}$ may simply add the two decimals from the question above.

$$0.025 + 0.00\overline{3} = 0.028\overline{3}$$

Problem #5

$$\frac{1}{99} \qquad\qquad \frac{1}{98}$$

Directions

- Use any method to calculate a decimal expression for each fraction. Continue until the expression repeats or you reach at least eight decimal places. Describe any patterns that you see.
- Find decimal expressions for other *99ths* fractions. Explain your thinking.
- Predict and test decimal expressions for *999ths* and *9999ths*, etc.
- Predict and test decimal expressions for *990ths*, *9900ths*. Etc. Explain your thinking.
- Carry out some similar investigations beginning with the fraction $\frac{1}{98}$.

Diving Deeper

- Investigate other denominators that lead to interesting decimal expressions. Here are a few that you could try: 49, 81, 891, 998, 9801, 998001.
- Find your own interesting denominators to explore!

Conversation Starters for #5

What do you notice? What do you wonder?

I notice a pattern for the *99ths* that reminds me of a pattern for the *9ths*.

I notice that $\frac{1}{99}$ is a little greater than $\frac{1}{100}$.

I notice a pattern for the 98ths that looks like it could keep going for a long time.

I notice some doubling patterns in the *98ths* decimals.

I wonder when (or if) the decimal for $\frac{1}{98}$ begins to repeat.

I wonder if some fractions have decimals with tripling patterns.

I wonder what happens with the decimals for the *97ths*.

I notice that 9, 99, 999, etc. are all 1 less than a power of 10.

I wonder what happens with denominators that are 2 less than a power of 10.

I wonder what happens with denominators that are 1 *greater* than a power of 10.

I notice a connection between the number of 9s in the denominators and the number of digits that repeat in the decimal.

I wonder what causes all of these patterns.

I wonder if I could find a fraction for any repeating decimal pattern.

Solutions for #5

Decimal expressions for $\frac{1}{99}$ *and* $\frac{1}{98}$

$$\frac{1}{99} = 0.\overline{01} \qquad \frac{1}{98} = \ 0.0102040816\ \ldots$$

Students may notice a doubling pattern in the second decimal. This pattern continues, though once the numbers in the doubling pattern have more than two digits, the decimal pattern is obscured by the effect of "carrying" digits over to neighboring place values. Students may be interested to know that the period of the decimal is 42. (The tenths digit, 0, is not part of the repeating pattern.)

Decimal expressions for other 99ths *fractions*

The decimals for the *99ths* fractions are formed by repeating the numerator. (Notice how similar this is to the *9ths* pattern!) For example, $\frac{47}{99} = 0.\overline{47}$. If the numerator has one digit, then the first digit in the repeating block is 0. For example, $\frac{5}{99} = 0.\overline{05}$. This pattern is caused by adding $0.\overline{01}$ whenever you increase the numerator by 1.

Decimal expressions for $\frac{1}{999}$ *and* $\frac{1}{9999}$

$$\frac{1}{999} = 0.\overline{001} \qquad \frac{1}{9999} = \ 0.\overline{0001}\ \ldots$$

These results are natural extensions of the patterns for $\frac{1}{9}$ and $\frac{1}{99}$. They lead to corresponding conclusions for the other *999ths* and *9999ths* fractions: the numerator is repeated to form the decimal, inserting zeros as needed to make the number of digits in the repeating block equal to the number of digits in the denominator. Examples:

$$\frac{47}{999} = 0.\overline{047} \qquad \frac{5378}{9999} = 0.\overline{5378}$$

Decimal expressions for 990ths*,* 9900ths*, etc.*

The decimal expressions for the 990ths look the same as those for the 99ths except that the digits are shifted one place value to the right, leaving a 0 in the tenths place. This happens because $\frac{n}{990}$ is one tenth of $\frac{n}{99}$. Similar comments apply to *9900ths*, except that the digits are shifted two places to the right. Examples:

$$\frac{37}{99} = 0.\overline{37} \qquad \frac{37}{990} = 0.0\overline{37} \qquad \frac{37}{9900} = 0.00\overline{37}$$

Extending the exploration of $\frac{1}{98}$

Students can extend their investigations in many directions. Some possibilities:

- Look at other 98ths fractions*. For example:

$$\frac{1}{98} = 0.01020408 \dots$$

$$\frac{2}{98} = 0.02040816 \dots$$

$$\frac{3}{98} = 0.03061224 \dots$$

- Multiply the denominator by powers of ten. For example:

$$\frac{17}{98} = 0.17346938 \dots$$

$$\frac{17}{980} = 0.017346938 \dots$$

$$\frac{17}{9800} = 0.0017346938 \dots$$

- Look at denominators that are 2 less than a power of 10. For example:

$$\frac{1}{98} = 0.01020408 \dots$$

$$\frac{1}{998} = 0.001002004008 \dots$$

$$\frac{1}{9998} = 0.0001000200040008 \dots$$

- Keep decreasing the denominator by 1. For example:

$$\frac{1}{98} = 0.01020408 \dots$$

$$\frac{1}{97} = 0.01030927 \dots$$

$$\frac{1}{96} = 0.01041666 \dots$$

Students should describe the patterns that they see. In some cases, they may also be able to explain what causes the patterns. (In other cases, the patterns are probably best explained using algebraic techniques that they will learn in a few years.)

**Note*: The decimals on this page are written without repeating bars, because the periods are too long to show the full repeating blocks of digits.

Problem #6

$$\frac{2}{3} = 0.\overline{6} \qquad \frac{2}{11} = 0.\overline{18}$$

$$0.5\overline{6} \qquad 0.6\overline{81}$$

Directions

- Use the equations in the top row to help you find fractions for the repeating decimals in the second row. Explain your thinking.
- Find another strategy for each problem (not necessary using the equations in the top row).

Conversation Starters for #6

What do you notice? What do you wonder?

I notice that the decimals in the top row have only repeating digits, but the decimals in the bottom row have a digit that does not repeat.

I notice that the digits 1 and 8 look like they trade places under the decimal bars on the right side.

I notice that the tenths digit is the only one that changes from each top decimal to the bottom one.

I notice that the first decimal decreases and the second one increases.

I wonder what the decimals for the other *11ths* fractions look like.

I wonder if would help to find a way to separate the non-repeating and repeating parts of the decimals.

I wonder if it would help to shift the place values so that the non-repeating part of each decimal is to the left of the decimal point.

Solutions for #6

Fractions for the repeating decimals

$$0.5\overline{6} = \frac{17}{30} \qquad 0.6\overline{81} = \frac{15}{22}$$

Sample strategies for $0.5\overline{6}$

$$0.5\overline{6} = 0.\overline{6} - 0.1 = \frac{2}{3} - \frac{1}{10} = \frac{20}{30} - \frac{3}{30} = \frac{17}{30}$$

$$0.5\overline{6} = 0.5 + 0.0\overline{6} = \frac{1}{2} + \frac{6}{90} = \frac{15}{30} + \frac{2}{30} = \frac{17}{30}$$

$$0.5\overline{6} = \frac{1}{10} \cdot 5.\overline{6} = \frac{1}{10} \cdot 5\frac{2}{3} = \frac{1}{10} \cdot \frac{17}{3} = \frac{17}{30}$$

$$0.5\overline{6} = 0.\overline{5} + 0.0\overline{1} = \frac{5}{9} + \frac{1}{90} = \frac{50}{90} + \frac{1}{90} = \frac{51}{90} = \frac{17}{30}$$

Other strategies are possible.

Sample strategies for $0.6\overline{81}$

$$0.6\overline{81} = 0.\overline{18} + 0.5 = \frac{2}{11} + \frac{1}{2} = \frac{4}{22} + \frac{11}{22} = \frac{15}{22}$$

$$0.6\overline{81} = 0.6 + 0.0\overline{81} = \frac{3}{5} + \frac{81}{990} = \frac{3}{5} + \frac{9}{110} = \frac{66}{110} + \frac{9}{110} = \frac{75}{110} = \frac{15}{22}$$

$$0.6\overline{81} = \frac{1}{10} \cdot 6.\overline{81} = \frac{1}{10} \cdot 6\frac{9}{11} = \frac{1}{10} \cdot \frac{75}{11} = \frac{75}{110} = \frac{15}{22}$$

Other strategies are possible.

Stage 3

In Problem #7, students investigate a fraction whose decimal expression repeats with a very large period. This process helps them solidify and extend concepts from Problem #3. It also prepares them to make an important discovery: every quotient of (non-zero) whole numbers—and thus every simple fraction—has a decimal expression that either terminates or repeats. This discovery lies at the heart of the definition of a *rational number*.

In Problem #8, students explore decimal expressions for fractions with denominators of 14. These expressions bring new patterns into play and offer student opportunities to develop new strategies as they work to understand what is happening.

In Problem #9, students integrate knowledge gained throughout the activity to develop general strategies for handling decimals with mixed non-repeating and repeating parts. In the process, they place the final piece of the puzzle in understanding the equivalence of the two definitions of a rational number. (See the Note at the end of the Solutions for #9.)

What students should know

- Understand concepts from Stages 1 and 2.

What students will learn

- Develop a deeper understanding of decimal patterns.
- Write any repeating decimal (including those with a non-repeating part) in fraction form.
- Begin to understand the relationship between the two equivalent definitions of a *rational number*.

Problem #7

$$17\overline{)1}$$

Directions

- Predict the maximum possible period of the decimal expression for $\frac{1}{17}$. Explain.
- Divide to find the decimal expression. Compare your answer to your prediction.
- Predict the decimal expressions for all of the *17ths* fractions. Explain your thinking.
- Explain why a quotient of two whole numbers (excluding 0) must always be a decimal that either terminates or repeats.

Diving Deeper

The decimal expressions for fractions with denominators of 7 and 17 have some important features in common. What features are these? Find other denominators whose decimals have these same features.

Conversation Starters for #7

What do you notice? What do you wonder?

I notice that none of the *17ths* fractions will simplify, because 17 is a prime number.

I wonder if the *17ths* decimals have the maximum possible period.

I notice that my calculator does not show enough digits to tell when or if the decimal repeats.

I notice that it is harder than before to do the calculations mentally.

I wonder if it would help to make an organized list of multiples of 17.

I notice that even when the digits in the quotient are the same, the remainders are not.

I notice that this reminds me of the process for finding *7ths* decimals.

I notice that I can use some of the same strategies that I used for finding *7ths* decimals.

I notice that when I increase the numerator by 1, the decimal increases by about 0.06.

I notice that there is never a remainder of 0, but all of the other possible remainders show up.

I notice that the number of fractions (not including $\frac{17}{17}$) is the same as the period of the decimal, which means that each digit in the repeating block has a "chance" to be the starting (tens) digit in the repeating block.

I notice that for each decimal, the number formed by the first eight digits plus the number formed by the last eight digits equals 999999!

This makes it easy to predict the last eight digits from the first eight digits.

I wonder if there is an easy way to predict the period of a repeating decimal without dividing.

Mathematicians do know of other ways to predict the period of a repeating decimal, but none of them are really easier than doing the division!

Solutions for #7

Predicting the maximum possible period

The maximum possible period is 16. There are seventeen possible remainders. If the decimal repeats, then 0 is not a remainder, leaving 16 possibilities. If you use all 16 of them, then the period will be 16, because there will be one digit in the repeating block for each remainder.

The decimal expression for $\frac{1}{17}$

$$\begin{array}{r} 0\,.0\;5\;8\;8\;2\;3\;5\;2\;9\;4\;1\;1\;7\;6\;4\;7 \\ 17\,\big)\overline{1.{}^{1}0{}^{10}0{}^{15}0{}^{14}0{}^{4}0{}^{6}0{}^{9}0{}^{5}0{}^{16}0{}^{7}0{}^{2}0{}^{3}0{}^{13}0{}^{11}0{}^{8}0{}^{12}0} \end{array}$$

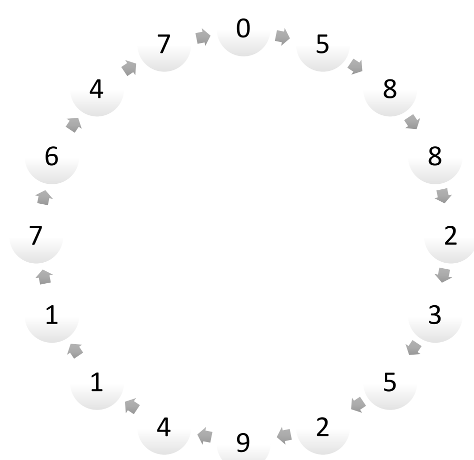
Therefore, the decimal expression is

$$\frac{1}{17} = 0.\overline{0588235294117647}$$

All 16 possible non-zero remainders appear, giving a repeating block of 16 digits.

The decimal expressions for the remaining 17ths *fractions*

Each time you add 1 to the numerator, choose the *tenths* digit from the cycle below that results in the next greater decimal. When two of the digits in the cycle are the same, look to the next digits in the cycle to determine which is smaller. Notice that each successive decimal will increase by approximately 0.06.



Note: Some students may notice that opposite numbers in the cycle always have a sum of 9! This ensures that the appropriate fractions have a sum of $0.\overline{9} = 1$.

$\frac{1}{17} = 0.0588235294117647$

$\frac{2}{17} = 0.1176470588235294$

$\frac{3}{17} = 0.1764705882352941$

$\frac{4}{17} = 0.2352941176470588$

$\frac{5}{17} = 0.2941176470588235$

$\frac{6}{17} = 0.3529411764705882$

$\frac{7}{17} = 0.4117647058823529$

$\frac{8}{17} = 0.4705882352941176$

$\frac{9}{17} = 0.5294117647058823$

$\frac{10}{17} = 0.5882352941176470$

$\frac{11}{17} = 0.6470588235294117$

$\frac{12}{17} = 0.7058823529411764$

$\frac{13}{17} = 0.7647058823529411$

$\frac{14}{17} = 0.8235294117647058$

$\frac{15}{17} = 0.8823529411764705$

$\frac{16}{17} = 0.9411764705882352$

Why quotients of counting numbers are always terminating or repeating decimals

There is a finite number of possible remainders. Either you get a remainder of 0 at some point in which case the decimal terminates, or some remainder eventually occurs a second time at which point the decimal begins to repeat.

Problem #8

$$\frac{1}{14} \quad \frac{2}{14} \quad \frac{3}{14} \quad \frac{4}{14} \quad \frac{5}{14} \quad \frac{6}{14} \quad \frac{7}{14} \quad \frac{8}{14} \quad \frac{9}{14} \quad \frac{10}{14} \quad \frac{11}{14} \quad \frac{12}{14} \quad \frac{13}{14} \quad \frac{14}{14}$$

Directions

- Find a decimal expression for each fraction. Use a variety of strategies, and describe them. Try to avoid long division as much as possible.
- Find and describe as many patterns as you can.

Diving Deeper

Carry out detailed explorations with other denominators: 13, 15, 16, 18, etc. Describe your strategies and any patterns that you discover.

Conversation Starters for #8

What do you notice? What do you wonder?

I notice that half of the fractions will simplify to *7ths* (which we have already explored).

I wonder if there will be other connections between *7ths* decimals and *14ths* decimals.

I notice that $\frac{1}{14}$ is half of $\frac{1}{7}$.

I notice that I can use the decimal for $\frac{1}{14}$ to build other decimals for the *14ths*.

I wonder if I can still use some of the strategies that worked for other fractions.

I notice decimals that have a combination of non-repeating and repeating parts.

I notice the same repeating blocks for the *14ths* that I saw in the *7ths*.

I wonder why this happens.

Solutions for #8

Decimal expressions for the 14ths

$\frac{1}{14} = 0.0\overline{714285}$

$\frac{2}{14} = 0.\overline{142857}$

$\frac{3}{14} = 0.2\overline{142857}$

$\frac{4}{14} = 0.\overline{285714}$

$\frac{5}{14} = 0.3\overline{571428}$

$\frac{6}{14} = 0.\overline{428571}$

$\frac{7}{14} = 0.5$

$\frac{8}{14} = 0.\overline{571428}$

$\frac{9}{14} = 0.6\overline{428571}$

$\frac{10}{14} = 0.\overline{714285}$

$\frac{11}{14} = 0.7\overline{857142}$

$\frac{12}{14} = 0.\overline{857142}$

$\frac{13}{14} = 0.9\overline{285714}$

$\frac{14}{14} = 0.\overline{9} = 1$

Sample strategies (alternatives to dividing)

- Simplify the fractions that have even numerators.
  This will give a *7ths* fraction whose decimal you already know.
- Find half of a *7ths* fraction.
  Example: Some students may find the decimal for $\frac{1}{14}$ by taking half of the decimal for $\frac{1}{7}$. (Some may even be able to do this mentally.)
- Subtract each digit from 9.
  For pairs of fractions that have a sum of 1, subtract each digit of the known decimal from 9 to find the other decimal in the pair.
  Example: If you subtract each digit of $0.0\overline{714285}$ (the decimal for $\frac{1}{14}$) from 9, you get $0.9\overline{285714}$, which is the decimal for $\frac{13}{14}$.
- Add two decimals that you have already calculated.
  Example: If you already know the decimals for $\frac{1}{14}$ and $\frac{2}{14}$, add them to get the decimal for $\frac{3}{14}$. $0.0\overline{714285} + 0.1\overline{428571} = 0.\overline{9285714}$. (This may be a little tricky, because you may have to add from left to right, looking ahead to anticipate any possible regrouping.)
- Use equivalent fractions and place value shifts.
  $$\frac{9}{14} = \frac{45}{70} = \frac{1}{10} \cdot \frac{45}{7} = \frac{1}{10} \cdot 6\frac{3}{7} = \frac{1}{10} \cdot 6.\overline{428571} = 0.6\overline{428571}$$
- Make use of patterns. (See next page.)

Some patterns

- The *14ths* decimals contain the same repeating patterns as the *7ths*, which seems reasonable, since 7 is a factor of 14. (This happens a lot with denominators that are composite numbers.)
- The decimals for fractions with odd numerators each have a single digit before the repeating pattern begins.
- Each succeeding decimal increases by approximately 0.07. (The first two digits of the decimals look like are counting up by sevens.)
- Each of the six repeating blocks for the 7ths occurs exactly once in the fractions with odd numerators.

Problem #9

0.324 $0.\overline{324}$ $0.3\overline{24}$ $0.32\overline{4}$

Directions

- Calculate a fraction for each decimal. Explain your thinking processes.
- Describe a general method for writing a repeating decimal in fraction form.
- Explain why every repeating or terminating decimal may be written as a *simple fraction* (a fraction with a whole number in the numerator and denominator).

Conversation Starters for #9

What do you notice? What do you wonder?

I notice that the first decimal terminates.

I notice that the first two decimals take very little work.

I notice that all of the decimals have the same digits, but the bar is in different places.

I wonder if there will be a simple pattern to the fractions that I get.

I notice that most of the strategies that I found in earlier questions are still helpful.

Solutions for #9

A fraction for 0.324

$$0.324 = \frac{324}{1000}$$

Read the result directly from the place value.

A fraction for $0.\overline{324}$

$$0.\overline{324} = \frac{324}{999} = \frac{12}{37}$$

Use the pattern discovered earlier in the exploration for denominators that are 1 less than a power of ten. Some students may observe that the answer makes sense because $\frac{324}{999}$ is slightly greater than $\frac{324}{1000}$.

A fraction for $0.3\overline{24}$

$$0.3\overline{24} = 0.3 + 0.0\overline{24} = \frac{3}{10} + \frac{24}{990} = \frac{3}{10} + \frac{8}{330} = \frac{99}{330} + \frac{8}{330} = \frac{107}{330}$$

$$0.3\overline{24} = \frac{1}{10} \cdot 3.\overline{24} = \frac{1}{10} \cdot 3\frac{24}{99} = \frac{1}{10} \cdot 3\frac{8}{33} = \frac{1}{10} \cdot \frac{107}{33} = \frac{107}{330}$$

Many other strategies are possible. Students may check their results by dividing.

A fraction for $0.32\overline{4}$

$$0.32\overline{4} = 0.32 + 0.00\overline{4} = \frac{32}{100} + \frac{4}{900} = \frac{288}{900} + \frac{4}{900} = \frac{292}{900} = \frac{73}{225}$$

$$0.32\overline{4} = \frac{1}{100} \cdot 32.\overline{4} = \frac{1}{100} \cdot 32\frac{4}{9} = \frac{1}{100} \cdot \frac{292}{9} = \frac{292}{900} = \frac{73}{225}$$

Many other strategies are possible. Students may check their results by dividing.

A method for writing a repeating decimal in fraction form

- Write the decimal as a sum of the repeating and non-repeating parts.
- Write each part as a fraction. For the repeating part of the decimal, use your knowledge of *9ths*, *99ths*, *999ths*, etc. combined with place value shifts.
- Add the fractions.

Other strategies are possible.

Why every repeating or terminating decimal may be written as a simple fraction

Every repeating or terminating decimal may be written as a simple fraction because the process described above applies to any repeating or terminating decimal and always results in a fraction with a whole number in the numerator and denominator.

Notes: In Problem #7, students explain why every simple fraction may be written as a repeating or terminating decimal. In Problem #9, students explain why every repeating or terminating decimal may be written as a simple fraction. Putting these two ideas together shows that the two definitions of a rational number are *equivalent*. In other words, if a number satisfies either condition, it automatically satisfies the other. (That is, both definitions mean the same thing!)

A rational number is a number that may be written as:

(1) A simple fraction (one that has a whole number numerator and denominator*).

(2) A terminating or repeating decimal.

*****The denominator may not equal 0. When students learn more about negative numbers, they will extend the definition of rational numbers to include *integers* as numerators and denominators. This allows for the possibility of certain negative values—the opposites of whole numbers.